Journal of Basics and Applied Sciences Research (JOBASR)
ISSN (print): 3026-9091, ISSN (online): 1597-9962
Volume 3(2) March 2025
DOI: https://dx.doi.org/10.4314/jobasr.v3i2.2

Notes on a Modified Exponential-Gamma Distribution: Its Properties and Applications

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ABSTRACT
Accurate Modeling and analysis of real-world data playa vital role across various fields, enabling better decision-making and predictions. While it is widely acknowledged that "all models are wrong, but some are useful." Nevertheless, researchers continuously develop, modify, extend, generalize and combine models with other distributionsto enhance accuracy and achieve significant progress. This paper introduces the Exponentiated Generalized new Exponential-Gamma distribution (EGnEG), a novel four parameters univariate continuous lifetime probability distribution that extends the new Exponential-Gamma distribution. The proposed distribution is named the Exponentiated Generalized new Exponential-Gamma distribution (EGnEG). Its survival and hazard rate functions of the distribution were derived and analyzed visually to understand its properties. Graphical representations of the probability density function (PDF), cumulative distribution function (CDF) and hazard rate function illustrate the distribution's behaviors across different parameter values.Additionally, Entropy measures and order statistic were determined to further assess its characteristics. The parameters of the EGnEG distribution were estimated using three different methods: Maximum Likelihood Method (MLE), Least Squares Estimation (LSE), and Cramer-Von-Mises Estimation (CVME). To assess its Goodness-of-fit, the distribution was applied to a real-life dataset and compared with that of some existing related distributions. The comparison based on the values of $-2logLik$, Akaike Information Criteria (AIC) Bayesian Information Criteria (BIC).The results from the dataset indicate that the Exponentiated Generalized New Exponential-Gamma (EGnEG) distribution out performs other competing distributions considered in the study. Therefore, this new distribution is recommended as a valuable alternative for modeling real life datasets, offering improved flexibility and accuracy in statistical modeling.

Keywords:
Probability, Exponential, Gamma, Exponentiated Exponential, Goodness of-fit

INTRODUCTION

In the field of applied sciences, including engineering, medical sciences, actuarial science, demography, public health, insurance, and finance, the reliability analysis and modeling of lifetime data play a crucial role. Accurate modeling of lifetime data is fundamental for predicting the failure rates of systems, products, or even individuals, enabling informed decision-making and resource allocation. To achieve this, statisticians have developed and refined various lifetime distributions (Amiru et al, 2025; Ibrahim et al, 2025; Olalekan et al, 2021) that can accommodate the diverse patterns of hazard functions observed in real-world phenomena.

Historically, the Exponential distribution has been a go-to model for lifetime data due to its simplicity and ease of use. However, it assumes a constant hazard rate, which may not be suitable for many practical applications where the failure rate changes over time. This limitation has prompted the development of more flexible models that can capture both increasing and decreasing hazard rates, such as the Exponential Geometric (EG) distribution (Adamidis & Loukas, 1998) and the Generalized Exponential (GE) distribution (Gupta & Kundu, 1999). These models, while more flexible, still face challenges in accommodating the complexities of real-world data.

How to cite this article: Suleman, I., Zakariyau, N. R., Oyegoke, O. A., Yahya, W. B., Amiru, F. M. and Umar, M. A. (2025). Notes on a Modified Exponential-Gamma Distribution: Its Properties and Applications. *Journal of Basics and Applied Sciences Research*, *3*(2), 12-26. https://dx.doi.org/10.4314/jobasr.v3i2.2

A significant advancement in this area is the introduction of the Exponential-Gamma (EG) distribution, first proposed by Ogunwale et al. (2019). This distribution, derived from a mixture of the Exponential and Gamma distributions, provides greater flexibility in modeling lifetime data with varying hazard functions. Further, a New Exponential-Gamma distribution (Umar & Yahya, 2021, 2019), and its extensions (Yahya & Umar, 2024 and 2025; Umar et al, 2019a & 2019b) that have the Exponential distribution, Lindley (and its extensions of) distributions (Gupta & Kundu, 2001; Lindley, 1958; Nadarajah et al, 2011; Zakayau et al, 2025) as special cases have improved its applicability, particularly in modeling data with non-constant hazard functions. The New Exponential-Gamma distribution has been shown to offer better fits to real-life data than the traditional Exponential distribution and some of its extensions.

However, there remains a gap in the available distributions that can capture more complex patterns in lifetime data while maintaining flexibility in their parameters. The existing models, although useful, still have limitations in dealing with specific types of data, particularly in accommodating complex hazard rate behaviors such as non-monotonic trends. This study addresses these limitations by extending the New Exponential-Gamma distribution to a more generalized form—Exponentiated Generalized New Exponential-Gamma (EGnEG) distribution.

The proposed EGnEG distribution introduces an additional layer of flexibility, allowing for more complex hazard functions and improved fitting to real-life datasets. By compounding the Exponentiated Exponential distribution with the New Exponential-Gamma distribution, the EGnEG distribution emerges as a four-parameter univariate continuous distribution. This new model is designed to provide a more accurate and adaptable framework for modeling lifetime data with varying failure rates, offering potential advantages in both theoretical and applied contexts.

The primary objective of this study is to derive the properties of the EGnEG distribution, explore its various special cases, and apply it to real datasets to demonstrate its effectiveness. Furthermore, this study will compare the performance of the EGnEG model with several conventional lifetime distributions, providing insights into its relative advantages for practical applications in diverse fields. Ultimately, the study aims to contribute to the ongoing development of more flexible and robust models for lifetime data analysis. The rest of the paper is organized as follows: Section 2 and 3 presents the model design, including the probability density function (PDF), cumulative distribution function (CDF), various properties, and expressions for the survival function and hazard rate function. Section 4 outlines the methods for parameter estimation. Section 5 showcases applications of the EGnEG model using real datasets, comparing its performance with several conventional lifetime distributions to demonstrate its relative superiority. Finally, Section 6 provides a concluding remark on the results and key findings.

MATERIALS AND METHODS

Model Design

The Lindley distribution (Lindley, 1958) is defined by its probability density function as;

$$f(x;\theta)\frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}; x>0, \theta>0 \quad (1)$$

This can be expressed as

$$pg_1(x;\theta)+qg_2(x;2,\theta) \quad (2)$$

where $g_1(x;\theta)=\theta e^{-\theta x}, g_2(x;2,\theta)=\theta^2 xe^{-\theta x}$ are the Exponential (θ) and Gamma $(2,\theta)$distributions respectively, $p=\frac{\theta}{\theta+1}$ and $q=1-p$.

The corresponding cumulative density function of the Lindley distribution is obtained as

$$F(x)=1-\left[1+\frac{\theta x}{\theta+1}\right]e^{-\theta x}; x>0, \theta>0 \quad (3)$$

The Exponentiated Lindley distribution is defined by Nadarajah et al (2011) as;

$$f(x;\theta,S)=\frac{S\theta^2}{\theta+1}(1+x)e^{-\theta x}\left(1-\left[1+\frac{\theta x}{\theta+1}\right]e^{-\theta x}\right)^{S-1}; x>0, \theta>0, S>0 \quad (4)$$

The Exponentiated Exponential distribution is defined (Gupta & Kundu, 2001) as;

$$f(x;\theta,S)=S\theta\left(1-e^{-\theta x}\right)^{S-1}e^{-\theta x}; x>0, S>0, \theta>0 \quad (5)$$

An Exponential-Gamma distribution (Ogunwale et al, 2019) is defined by its p.d.f as:

$$f(x;\alpha,\theta)=\frac{x^{\alpha-1}\theta^{\alpha+1}e^{-2\theta x}}{\Gamma(\alpha)};\ x>0, \alpha>0, \theta>0 \quad (6)$$

It is expressed as the product of Exponential and Gamma density functions.

That is, $f(x;\ \alpha,\theta)=f(x_1,x_2)=f(x_1)\cdot f(x_2)$.

The New Exponential-Gamma distribution is defined by Umar & Yahya (2021): as

$$f(x;\alpha,\theta)=\frac{\theta}{\theta+\Gamma(\alpha)}(\theta+\theta^{\alpha-1}x^{\alpha-1})e^{-\theta x};\ x>0, \alpha>0, \theta>0 \quad (7)$$

The distribution is expressed as a two components mixture of Exponential (θ) and Gamma (α,θ), generalizing the Lindley distribution in (1).

The corresponding cumulative distribution function is thus obtained as:

$$F(x;\alpha,\theta)=\int_0^x f(t;\alpha,\theta)dt \quad (8)$$

This implies thus, that

$$F(x;\alpha,\theta)=\frac{1}{\theta+\Gamma(\alpha)}\left[\theta\left(1-e^{-\theta x}\right)+\theta^\alpha L\right] \quad (9)$$

Where$L=\int_0^x t^{\alpha-1}e^{-\theta t}dt$

It can be easily verified that when $\alpha=1$, Exponential-Gamma distribution reduces to the Exponential distribution, and, when $\alpha=2$, the distribution reduces to the Lindley distribution (Lindley, 1958).

Alzaatreh et al. (2013), has introduced Beta Exponential - X family which has following cumulative density function (CDF) and probability density function (PDF),

$$F(x;\ \alpha,\beta,\lambda) = 1 - I_{[1-F(x)]^{\lambda}}[\lambda(\beta-1)+1,\alpha] \qquad (10)$$

$$f(x;\ \alpha,\beta,\lambda) = \frac{\lambda}{B(\alpha,\beta)}g(x)\left[[1-G(x)]^{\lambda\beta-1}[1-\{1-F(x)\}]^{\lambda}\right]^{\alpha-1} \qquad (11)$$

Where, I denote incomplete beta function. For $\beta = 1$, above CDF and PDF reduces to Exponentiated Generalized (EG) class of distribution with CDF and PDF as,

$$F(x) = \left[1-\{1-G(x)\}^{\beta}\right]^{\lambda} \qquad (12)$$

And,

$$f(x) = \beta\lambda g(x)[1-G(x)]^{\beta-1}\left[1-\{1-G(x)\}^{\beta}\right]^{\lambda-1} \qquad (13)$$

Telee et al., (2022) introduced Exponentiated Generalized Exponential Geometric (EGEG) Distribution which has a cumulative density function (CDF) and a probability density function (PDF), as

$$F(x) = \left[1-\left\{\frac{\theta e^{-\beta x}}{\{1-(1-\theta)e^{-\beta x}\}}\right\}^{\lambda}\right]^{\alpha} \qquad (14)$$

$$f(x) = \frac{\alpha\lambda\beta\theta e^{-\beta x}}{\{1-(1-\theta)e^{-\beta x}\}^{2}}\left[\frac{\theta e^{-\beta x}}{\{1-(1-\theta)e^{-\beta x}\}}\right]^{\lambda-1}\left[1-\left\{\frac{\theta e^{-\beta x}}{\{1-(1-\theta)e^{-\beta x}\}}\right\}^{\lambda}\right]^{\alpha-1} \qquad (15)$$

Exponentiated Generalized new Exponential Gamma (EGnEG) Distribution

An Exponentiated version of probability density function is most conveniently specified in terms of the cumulative distribution function (CDF) (Rather & Subramanian, 2019 and 2020, among others).

Here, we have used CDF of new Exponential-Gamma distribution function G(x) as the base line distribution function having CDF and PDF as

$$G(x) = \frac{1}{\theta+\Gamma(\alpha)}\left[\theta\left(1-e^{-\theta x}\right)+\theta^{\alpha}L\right] \qquad (16)$$

Where $L = \int_0^x t^{\alpha-1}e^{-\theta t}dt$

$$g(x) = \frac{\theta}{\theta+\Gamma(\alpha)}(\theta+\theta^{\alpha-1}x^{\alpha-1})e^{-\theta x};\ x>0, \alpha>0, \theta>0 \qquad (17)$$

Substituting the density function $g(x)$ in density function of Exponentiated Exponential X family (15) and (16), we get a new density function named as Exponentiated Generalized new Exponential Gamma (EGnEG) distribution. The distribution function and density function of proposed model EGnEG is given as

$$F(x) = \left[1-\left\{1-\frac{\left[\theta\left(1-e^{-\theta x}\right)+\theta^{\alpha}L\right]}{\theta+\Gamma(\alpha)}\right\}^{\beta}\right]^{\lambda} = \left[1-\left\{\frac{\left[\theta e^{-\theta x}+\Gamma(\alpha)-\theta^{\alpha}L\right]}{\theta+\Gamma(\alpha)}\right\}^{\beta}\right]^{\lambda} \qquad (18)$$

where $L = \int_0^x t^{\alpha-1}e^{-\theta t}dt$

$$f(x) = \frac{\beta\theta\lambda(\theta+\theta^{\alpha-1}x^{\alpha-1})e^{-\theta x}}{\theta+\Gamma(\alpha)}\left\{\frac{\left[\theta e^{-\theta x}+\Gamma(\alpha)-\theta^{\alpha}L\right]}{\theta+\Gamma(\alpha)}\right\}^{\beta-1}\left[1-\left\{\frac{\left[\theta e^{-\theta x}+\Gamma(\alpha)-\theta^{\alpha}L\right]}{\theta+\Gamma(\alpha)}\right\}^{\beta}\right]^{\lambda-1} \qquad (19)$$

where, $x>0, \alpha,\beta,\lambda, and\ \theta>0$. $\alpha,\beta, and\ \lambda$ Shape parameters. θ is scale parameter and $L = \int_0^x t^{\alpha-1}e^{-\theta t}dt$

Special cases:

Let X denotes the non-negative random variable with pdf given in equation (19). We can define some other sub models from the proposed model as:

1. For $\alpha = 1$, EGnEG reduces to the Exponentiated Generalized Exponential distribution as

$$F(x) = \left[1-\left\{1-\frac{\left[\theta\left(1-e^{-\theta x}\right)+\theta L\right]}{\theta+1}\right\}^{\beta}\right]^{\lambda} = \left[1-\left\{\frac{\left[\theta e^{-\theta x}+1-\theta L\right]}{\theta+1}\right\}^{\beta}\right]^{\lambda}$$

$$f(x) = \beta\theta\lambda e^{-\theta x}\left\{\frac{\left[\theta e^{-\theta x}+1-\theta L\right]}{\theta+1}\right\}^{\beta-1}\left[1-\left\{\frac{\left[\theta e^{-\theta x}+1-\theta L\right]}{\theta+1}\right\}^{\beta}\right]^{\lambda-1}$$

Where $L = \int_0^x e^{-\theta t}dt$

2. For $\alpha = 2$, EGnEG reduces to the Exponentiated Generalized Lindley distribution as

$$F(x) = \left[1-\left\{1-\frac{\left[\theta\left(1-e^{-\theta x}\right)+\theta^2 L\right]}{\theta+1}\right\}^{\beta}\right]^{\lambda} = \left[1-\left\{\frac{\left[\theta e^{-\theta x}+1-\theta^2 L\right]}{\theta+1}\right\}^{\beta}\right]^{\lambda}$$

$$f(x) = \frac{\beta\theta^2\lambda(1+x)e^{-\theta x}}{\theta+1}\left\{\frac{\left[\theta e^{-\theta x}+1-\theta^2 L\right]}{\theta+1}\right\}^{\beta-1}\left[1-\left\{\frac{\left[\theta e^{-\theta x}+1-\theta^2 L\right]}{\theta+1}\right\}^{\beta}\right]^{\lambda-1}$$

Where $L = \int_0^x te^{-\theta t}dt$

3. For $\alpha = 1,\ \lambda = 1$ and $\beta = 1$, EGnEG reduces to the Exponential distribution.
4. For $\alpha = 2,\ \lambda = 1$ and $\beta = 1$, EGnEG reduces to the Lindley distribution in (1) (Lindley, 1958).
5. For $\lambda = 1$ and $\beta = 1$,the proposed model reduces to the new Exponential Gamma distribution.

6. For $\alpha = 1$ and $\beta = 1$, EGnEG reduces to the Exponentiated Exponential distribution (Gupta & Kundu, 2001).
7. For $\alpha = 2$ and $\beta = 1$, EGnEG reduces to the Exponentiated Lindley distribution (Nadarajah et al, 2011).
8. For $\beta = 1$, the proposed model reduces to Exponentiated new Exponential Gamma Distribution (Zakariyau et. al. , 2025)as,

$$f(x) = \frac{\theta\lambda(\theta+\theta^{\alpha-1}x^{\alpha-1})e^{-\theta x}}{\theta+\Gamma(\alpha)}\left[1-\left\{\frac{[\theta e^{-\theta x}+\Gamma(\alpha)-\theta^{\alpha}L]}{\theta+\Gamma(\alpha)}\right\}\right]^{\lambda-1} \quad (20)$$

$$F(x) = \left[1-\left\{\frac{[\theta e^{-\theta x}+\Gamma(\alpha)-\theta^{\alpha}L]}{\theta+\Gamma(\alpha)}\right\}\right]^{\lambda}, \text{ where } L = \int_0^x t^{\alpha-1}e^{-\theta t}dt \quad (21)$$

The density plot of the proposed model EGnEG at various values of α, θ, β and λ is shown in figure 1. It can be observed that:

1. Effect of α (Shape Parameter):
   - When α increases, it generally shifts the peak of the PDF to the right and increases the spread of the distribution. This means the distribution becomes more spread out (e.g., more likely to take larger values).
   - For smaller values of α, the PDF will peak sharply and decay more quickly.
   - In the **CDF**, a higher α will cause the cumulative probability to rise more gradually over the range of X.
2. Effect of β (Exponentiation Parameter):
   - A smaller β results in a sharper peak near zero (indicating higher probability density at smaller values of X).
   - As β increases, the distribution becomes more spread out, with the PDF decaying more slowly and the CDF rising more slowly.
   - For example, when $\beta = 2$, the CDF will tend to approach 1 more gradually, meaning the probability accumulates more slowly.
3. Effect of λ (Exponentiation Factor):
   - A higher λ increases the tail heaviness of the distribution, causing the PDF to decay more slowly at larger X values. This means the distribution has a longer tail, and the random variable X is more likely to take large values.
   - In the CDF, a higher λ will cause the function to rise more slowly, implying it takes a longer X-range to reach a cumulative probability of 1.
4. Effect of θ (Scale Parameter):
   - Larger values of θ spread out the distribution and decrease the peak of the PDF. It also shifts the distribution rightward, making larger values of X more probable.
   - In the CDF, increasing θ will cause the curve to rise more gradually, meaning it takes longer for the cumulative probability to reach 1.

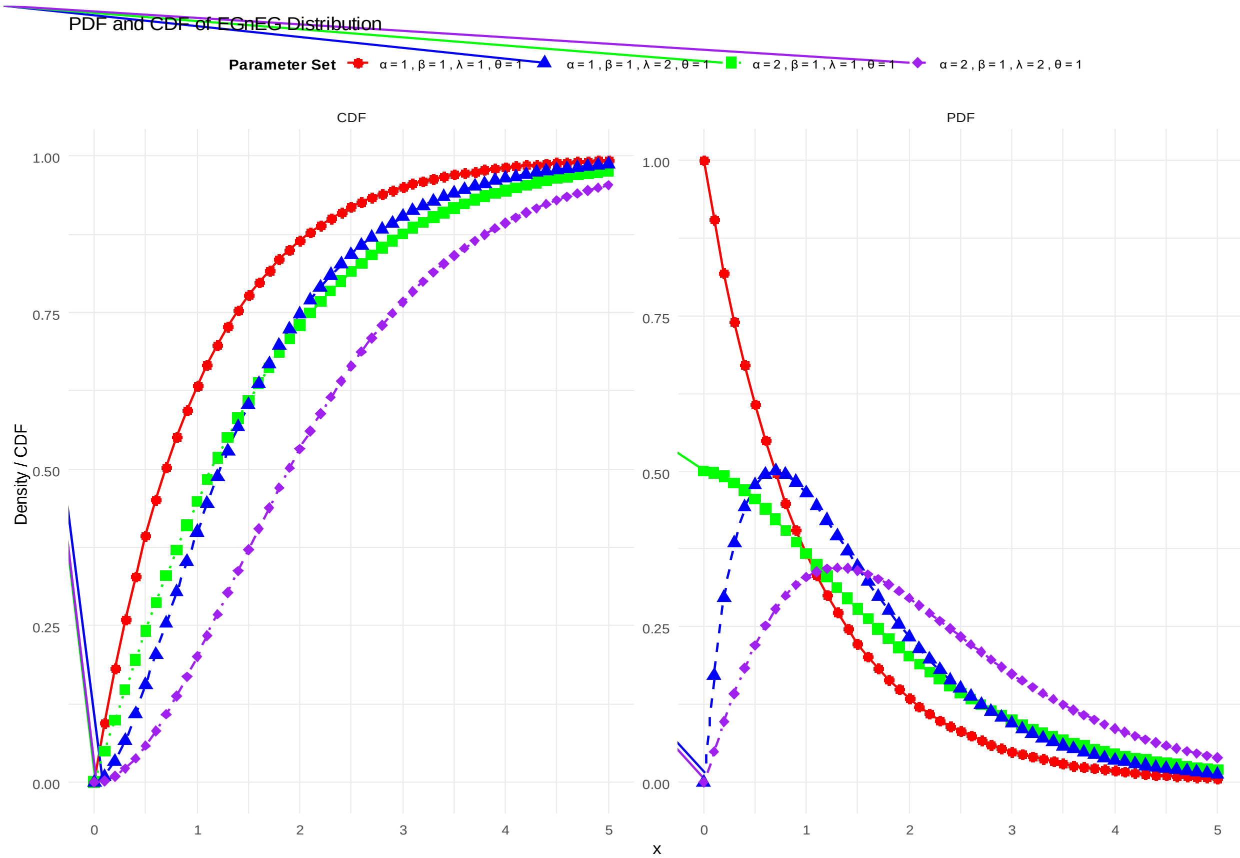

Figure 1: The density plot of theEGnEG distribution at various values of α, θ, β and λ

Specific Effects on the Plot:

1. **For $\alpha = 1, \beta = 1, \lambda = 1$ and $\theta = 1$:**
   - This combination will gives the **Exponential distribution**. The PDF will have a simple exponential decay, and the CDF will rise steadily as x increases.
2. **For $\alpha = 2, \beta = 1, \lambda = 1$ and $\theta = 1$:**
   - This combination will give the **Lindley distribution**. The PDF will be more spread out compared to the Exponential distribution, and the CDF will rise more slowly, indicating a more gradual accumulation of probability over x.
3. **For $\alpha = 1, \beta = 1, \lambda = 2$ and $\theta = 1$:**
   - Increasing λ causes the distribution to have a heavier tail. The PDF will decay slower, and the CDF will increase more gradually, showing that larger values of x are more probable.
4. **For $\alpha = 2, \beta = 1, \lambda = 2$ and $\theta = 1$:**
   - With both α and λ increasing, the PDF becomes more spread out, with a slower decay at the right tail. The CDF will rise more gradually, as the probability accumulates more slowly with respect to x.

From the density plot, it is clear that density plot of EGnEG can take different shapes.For smaller α, the PDF peaks quickly and decays more rapidly while the CDF rises more sharply. As $\alpha$and β increase, the peak shifts to the right, and the tail becomes more spread out, with slower decay. A higher λ also leads to a slower decay in the tail. Larger α, λ, and θ values will cause the CDF to rise more gradually, indicating that values of x take longer to accumulate probability.It can be observed that the **PDF** for higher values of αor λ spreads out, and the **CDF** for these same values rises more gradually.

Statistical Properties

Major characteristics of the proposed model EGNEG are derived in this section.

1 Survival rate function

The survival function is defined as the probability of an event not failing before specified time t. Survival function of EGnEG is given as

$$S(x) = 1 - F(x) \quad (22)$$

$$S(x) = 1 - \left[1 - \left\{1 - \frac{[\theta(1-e^{-\theta x}) + \theta^{\alpha}L]}{\theta + \Gamma(\alpha)}\right\}^{\beta}\right]^{\lambda} = \left[1 - \left\{\frac{[\theta e^{-\theta x} + \Gamma(\alpha) - \theta^{\alpha}L]}{\theta + \Gamma(\alpha)}\right\}^{\beta}\right]^{\lambda} \quad (23)$$

The **survival function $S(x)$** represents the probability that the random variable X takes a value greater than x. In other words, $S(x)$ gives us the likelihood that X "survives" beyond a particular threshold x.

- **As $x \to 0$**: The survival function $S(x)$ is expected to be close to 1, as the probability that X exceeds 0 is very high (since X starts from 0).
- **As x increases**: The survival function will decrease because the probability that X exceeds x decreases as x increases. It will approach 0 as $x \to \infty$.

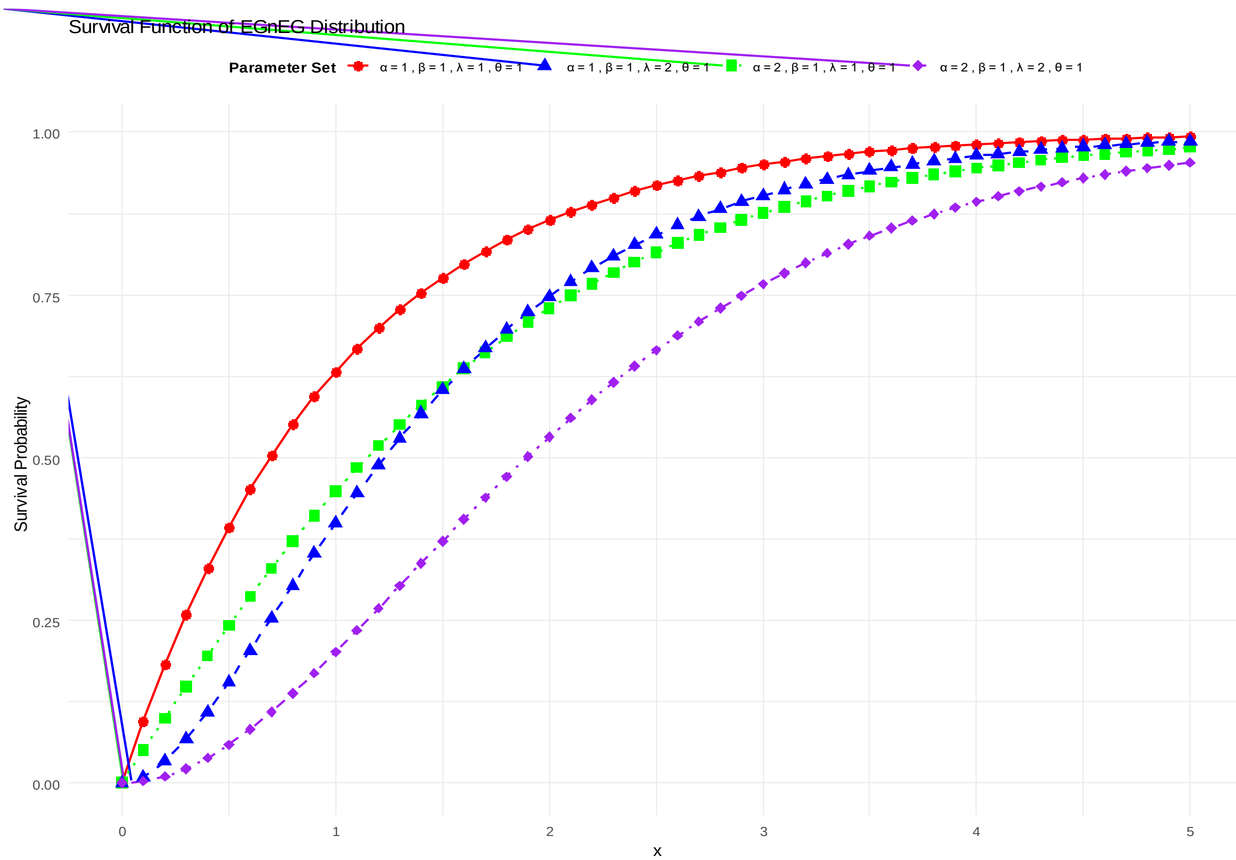

Figure 2: The Survival rate function plot of theEGnEG distribution at various values of α, θ, β and λ

For small x, the survival function starts close to 1. This is because the probability that X exceeds a small value is high.As x increases, the survival function decreases, as the probability of X exceeding x becomes smaller. The rate of decrease depends on the model parameters α, β, and λ. Higher values of these parameters lead to slower decay in the survival function (indicating that the random variable Xhas a longer "tail"). Larger values of α, β, or λmakes the survival function decay more slowly. This means that the "tail" of the distribution is heavier, and there is a higher chance that Xtakes larger values. This plot visually captures how the survival probability changes with different parameter values and show you the impact of the EGnEG distribution's tail behavior.

Hazard rate function

The hazard function is the defined as the instant failure rate at a given time t. The hazard function $h(x)$ of the proposed model is given as

$$h(x) = \frac{f(x)}{1-F(x)} = \frac{f(x)}{S(x)} \quad (24)$$

$$= \frac{\beta\theta\lambda(\theta+\theta^{\alpha-1}x^{\alpha-1})e^{-\theta x}}{\theta+\Gamma(\alpha)}\left\{\frac{[\theta e^{-\theta x}+\Gamma(\alpha)-\theta^{\alpha}L]}{\theta+\Gamma(\alpha)}\right\}^{\beta-1}\left[1 - \left\{\frac{[\theta e^{-\theta x}+\Gamma(\alpha)-\theta^{\alpha}L]}{\theta+\Gamma(\alpha)}\right\}^{\beta}\right]^{\lambda-1}\left[\left[1 - \right.\right.$$

$$\left\{\frac{[\theta e^{-\theta x}+\Gamma(\alpha)-\theta^{\alpha}L]}{\theta+\Gamma(\alpha)}\right\}^{\beta}\Bigg]^{\lambda}\Bigg]^{-1} \qquad (25)$$

The hazard function gives the instantaneous rate of failure (or hazard) at time x. It can provide insight into the shape and behavior of the distribution:

- If $h(x)$ is constant over time, this indicates a memoryless or exponential distribution.
- If $h(x)$ is increasing over time, it indicates that the failure rate increases as x increases, which is characteristic of distributions with "heavy" tails or distributions where the likelihood of an event increases over time.
- If $h(x)$ is decreasing over time, it indicates that the failure rate decreases as x increases, which is common in systems where the risk of failure reduces as time progresses.

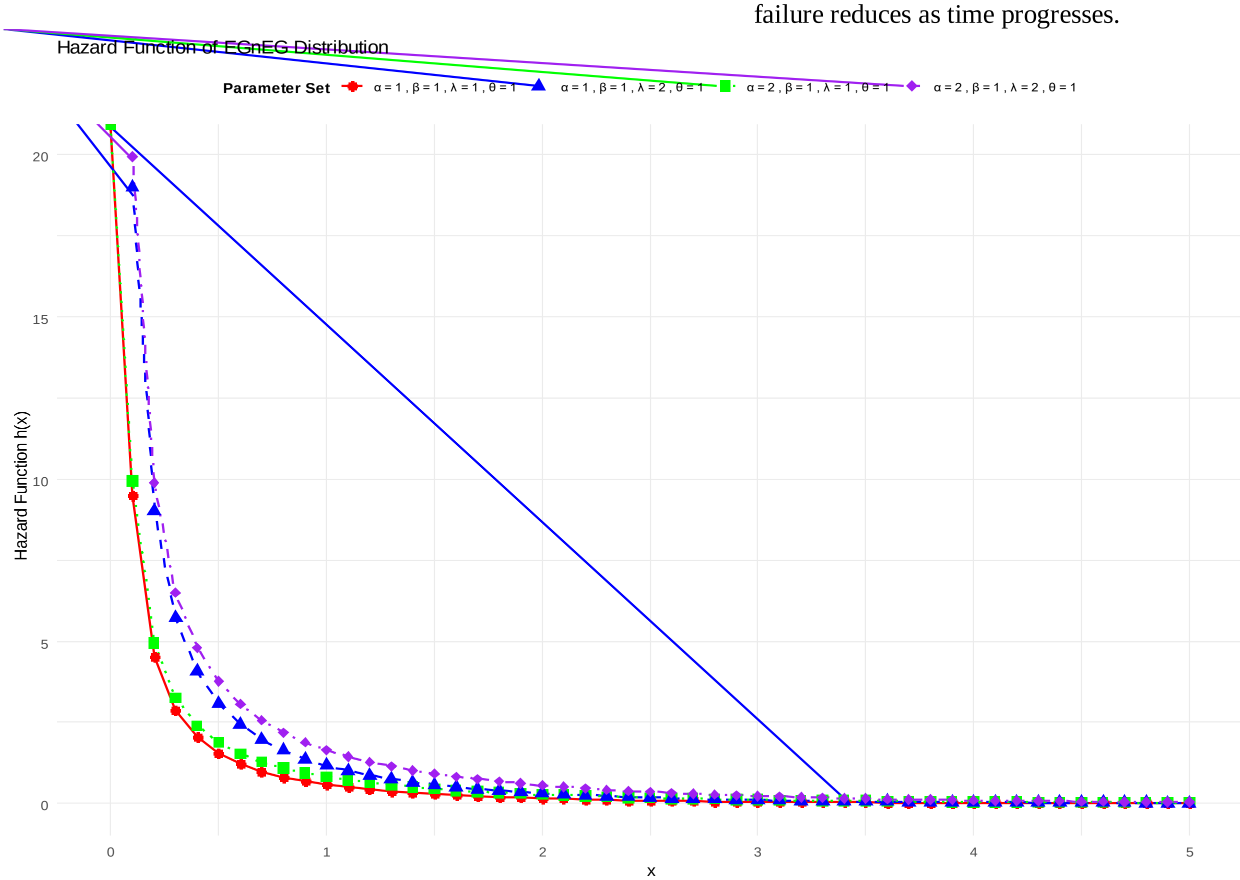

Figure 3: The Hazard rate function plot of theEGnEG distribution at various values of α, θ, β and λ

Increasing Hazard Function: If the hazard function $h(x)$ increases as x increases, it indicates that the likelihood of "failure" increases as time goes on. This is often the case in distributions that model systems with aging or wear-out processes, where the risk increases over time.

Decreasing Hazard Function: If $h(x)$ decreases as x increases, it implies that the "failure rate" decreases over time. This may indicate systems that become more stable or resilient as time progresses.

Flat Hazard Function: If $h(x)$ is relatively flat, it suggests a **memoryless** process, which is a characteristic of the **exponential distribution**.

By plotting the hazard function for different parameter values α, β, λ, and θ, gives insights into how these parameters influence the failure rate over time.

Asymptotic properties

To check whether the model is uni-modal or not, some of its asymptotic properties are studies. For this, we have found the limiting values of density function $f(x)$ in equation (19) at $x = 0$ and $x = \infty$. That is, for $x \rightarrow 0$.

$$\lim_{x\to 0} f(x;\ \alpha,\beta,\lambda,\theta)$$

$$= \lim_{x\to 0}\left[\frac{\beta\theta\lambda(\theta+\theta^{\alpha-1}x^{\alpha-1})e^{-\theta x}}{\theta+\Gamma(\alpha)}\left\{\frac{[\theta e^{-\theta x}+\Gamma(\alpha)-\theta^{\alpha}L]}{\theta+\Gamma(\alpha)}\right\}^{\beta-1}\left[1 - \left\{\frac{[\theta e^{-\theta x}+\Gamma(\alpha)-\theta^{\alpha}L]}{\theta+\Gamma(\alpha)}\right\}^{\beta}\right]^{\lambda-1} = 0\right]$$

Similarly for $x \rightarrow \infty$

$$\lim_{x\to \infty} f(x;\ \alpha,\beta,\lambda,\theta)$$

$$= \lim_{x\to\infty}\left[\frac{\beta\theta\lambda(\theta+\theta^{\alpha-1}x^{\alpha-1})e^{-\theta x}}{\theta+\Gamma(\alpha)}\left\{\frac{[\theta e^{-\theta x}+\Gamma(\alpha)-\theta^{\alpha}L]}{\theta+\Gamma(\alpha)}\right\}^{\beta-1}\left[1-\left\{\frac{[\theta e^{-\theta x}+\Gamma(\alpha)-\theta^{\alpha}L]}{\theta+\Gamma(\alpha)}\right\}^{\beta}\right]^{\lambda-1} = 0\right]$$

Since the limiting values of $f(x)\, for\; x \to 0\; and\; for\; x \to \infty$ are 0 confirms that the proposed model EGnEG is uni-modal.

Entropy Measures

Shannon Entropy

Shannon Entropy is a measure of uncertainty or information content of a random variable. The Shannon Entropy $H(X)$ for a continuous random variable with probability density function $f(x)$ is given by:

$$H(X) = -\int_0^{\infty} f(x)\ln\big(f(x)\big)\,dx \qquad (26)$$

For the EGnEG distribution, we substitute the PDF of the distribution $f(x)$in (19) into this formula in (26) as:

$$H(X) = -\int_0^{\infty}\frac{\beta\theta\lambda(\theta+\theta^{\alpha-1}x^{\alpha-1})e^{-\theta x}}{\theta+\Gamma(\alpha)}\left\{\frac{[\theta e^{-\theta x}+\Gamma(\alpha)-\theta^{\alpha}L]}{\theta+\Gamma(\alpha)}\right\}^{\beta-1}\left[1-\left\{\frac{[\theta e^{-\theta x}+\Gamma(\alpha)-\theta^{\alpha}L]}{\theta+\Gamma(\alpha)}\right\}^{\beta}\right]^{\lambda-1}\ln\left(\frac{\beta\theta\lambda(\theta+\theta^{\alpha-1}x^{\alpha-1})e^{-\theta x}}{\theta+\Gamma(\alpha)}\left\{\frac{[\theta e^{-\theta x}+\Gamma(\alpha)-\theta^{\alpha}L]}{\theta+\Gamma(\alpha)}\right\}^{\beta-1}\left[1-\left\{\frac{[\theta e^{-\theta x}+\Gamma(\alpha)-\theta^{\alpha}L]}{\theta+\Gamma(\alpha)}\right\}^{\beta}\right]^{\lambda-1}\right)dx \qquad (27)$$

While this is quite complex, it is solvable through numerical integration techniques for specific values of the parameters α, β, λ, and θ.

Renyi Entropy

The Renyi entropy of order α is a generalization of Shannon entropydefined as:

$$H_{\alpha}(X) = \frac{1}{1-\alpha}\ln\left(\int_0^{\infty} f(x)^{\alpha}\,dx\right) \qquad (28)$$

For the EGnEG distribution, we substitute the PDF $f(x)$in (19) into the above formula in (28). For $\alpha \neq 1$, the Renyi entropy can be computed as:

$$H_{\alpha}(X) = \frac{1}{1-\alpha}\ln\left(\int_0^{\infty}\left(\frac{\beta\theta\lambda(\theta+\theta^{\alpha-1}x^{\alpha-1})e^{-\theta x}}{\theta+\Gamma(\alpha)}\left\{\frac{[\theta e^{-\theta x}+\Gamma(\alpha)-\theta^{\alpha}L]}{\theta+\Gamma(\alpha)}\right\}^{\beta-1}\left[1-\left\{\frac{[\theta e^{-\theta x}+\Gamma(\alpha)-\theta^{\alpha}L]}{\theta+\Gamma(\alpha)}\right\}^{\beta}\right]^{\lambda-1}\right)^{\alpha}dx\right) \qquad (29)$$

Again, this integral requires numerical techniques to compute in practice. Where L is as defined earlier.

Order Statistic for the EGnEG distribution

Order statistics refer to the distribution of the k^{th} smallest value in a sample of size n. The probability density function for the k^{th} order statistic $X_{(k)}$ for a random sample from the EGnEG distribution is given by:

$$f_{X_{(k)}}(x) = \frac{n!}{(k-1)!(n-k)!}[F_X(x)]^{k-1}[1-F_X(x)]^{n-k}f_X(x) \qquad (30)$$

where:

- $F_X(x)$ is the cumulative distribution function (CDF) of EGnEG distribution.
- $f_X(x)$ is the probability mass function (PDF) of EGnEG distribution.

Thus, the PDF of the k^{th}order statistic for the EGnEG distribution is given as:

$$f_{X_{(k)}}(x) = \frac{n!}{(k-1)!(n-k)!}\left[\left[1-\left\{\frac{[\theta e^{-\theta x}+\Gamma(\alpha)-\theta^{\alpha}L]}{\theta+\Gamma(\alpha)}\right\}^{\beta}\right]^{\lambda}\right]^{k-1}\left[1-\left[1-\left\{\frac{[\theta e^{-\theta x}+\Gamma(\alpha)-\theta^{\alpha}L]}{\theta+\Gamma(\alpha)}\right\}^{\beta}\right]^{\lambda}\right]^{n-k} f_X(x) \qquad (31)$$

This is another integral that would require numerical methods to evaluate for specific values of the parameters and sample size.

Parameter Estimation

Parameters of the new distribution are estimated using the three commonly used estimation methods; Maximum likelihood estimators (MLE), Cramer-Von-Mises (CVM) and Least-squares (LSE) methods.

Maximum Likelihood Estimation (MLE)

In this section, we have presented the ML estimators (MLE's) of the EGnEG distribution.

Let $x_i, i = 1, 2, 3, \dots n$, be a random sample of size 'n' from the Exponentiated Generalized New Exponential-Gamma distribution. The likelihood function L of x is defined as;

$$L = \prod_{i}^{n} f(x;\alpha,\beta,\lambda,\theta)$$

$$= \prod_{i}^{n}\left[\frac{\beta\theta\lambda(\theta+\theta^{\alpha-1}x^{\alpha-1})e^{-\theta x}}{\theta+\Gamma(\alpha)}\left\{\frac{[\theta e^{-\theta x}+\Gamma(\alpha)-\theta^{\alpha}L]}{\theta+\Gamma(\alpha)}\right\}^{\beta-1}\left[1-\left\{\frac{[\theta e^{-\theta x}+\Gamma(\alpha)-\theta^{\alpha}L]}{\theta+\Gamma(\alpha)}\right\}^{\beta}\right]^{\lambda-1}\right]$$

$$= \frac{(\beta\lambda\theta)^n}{\big(\theta+\Gamma(\alpha)\big)^n}\cdot e^{-\theta\sum_i^n x_i}\sum_i^n[\theta + \theta^{\alpha-1}(x_i)^{\alpha-1}]\sum_i^n\left\{\frac{[\theta e^{-\theta x_i}+\Gamma(\alpha)-\theta^{\alpha}L]}{\theta+\Gamma(\alpha)}\right\}^{\beta-1} \times \sum_i^n\left[1-\left\{\frac{[\theta e^{-\theta x_i}+\Gamma(\alpha)-\theta^{\alpha}L]}{\theta+\Gamma(\alpha)}\right\}^{\beta}\right]^{\lambda-1}$$

Taking natural log likelihood is thus obtained as

$$\ell(\alpha,\beta,\lambda,\theta/\underline{x}) = nln(\beta\theta\lambda) - nln(\theta+\Gamma(\alpha)) - \theta\sum_i^n x_i + (\beta-1)\sum_i^n ln\left[\frac{[\theta e^{-\theta x}+\Gamma(\alpha)-\theta^\alpha L]}{\theta+\Gamma(\alpha)}\right] + \sum_i^n ln\,[\theta+\theta^{\alpha-1}(x_i)^{\alpha-1}] + (\lambda-1)\sum_i^n ln\left[1-\left\{\frac{[\theta e^{-\theta x_i}+\Gamma(\alpha)-\theta^\alpha L]}{\theta+\Gamma(\alpha)}\right\}^\beta\right] \quad (32)$$

Differentiating (32) with respect to the parameters $\alpha,\beta,\lambda\ and\ \theta$, we get

$$\frac{\partial\ell}{\partial\alpha} = -\frac{n\Gamma'(\alpha)}{\theta+\Gamma(\alpha)} + \sum_{i=1}^n\left[\frac{(\theta x_i)^{\alpha-1}\ln(\theta x_i)}{\theta+(\theta x_i)^{\alpha-1}}\right] + (\beta-1)\sum_{i=1}^n\left[\frac{(\theta-\theta e^{-\theta x_i}+\theta^\alpha L)\Gamma'(\alpha)-(\theta+\Gamma(\alpha))\theta^\alpha L ln\theta}{(\theta+\Gamma(\alpha))(\theta-\theta e^{-\theta x_i}+\theta^\alpha L)}\right]$$
$$-\beta(\lambda-1)\sum_{i=1}^n\left[\left[1-\left\{\frac{\theta e^{-\theta x_i}+\Gamma(\alpha)-\theta^\alpha L}{\theta+\Gamma(\alpha)}\right\}^\beta\right]^{-1}\left[\frac{\theta e^{-\theta x_i}+\Gamma(\alpha)-\theta^\alpha L}{\theta+\Gamma(\alpha)}\right]^{\beta-1}\left[\frac{(\theta-\theta e^{-\theta x_i}+\theta^\alpha L)\Gamma'(\alpha)-(\theta+\Gamma(\alpha))\theta^\alpha L ln\theta}{(\theta+\Gamma(\alpha))(\theta-\theta e^{-\theta x_i}+\theta^\alpha L)}\right]\right]$$

$$\frac{\partial\ell}{\partial\beta} = \frac{n}{\beta} + \sum_{i=1}^n ln\left[\frac{\theta e^{-\theta x_i}+\Gamma(\alpha)-\theta^\alpha L}{\theta+\Gamma(\alpha)}\right]$$
$$-(\lambda-1)\sum_{i=1}^n\left[1-\left\{\frac{\theta e^{-\theta x_i}+\Gamma(\alpha)-\theta^\alpha L}{\theta+\Gamma(\alpha)}\right\}^\beta\right]^{-1}\left[\frac{\theta e^{-\theta x_i}+\Gamma(\alpha)-\theta^\alpha L}{\theta+\Gamma(\alpha)}\right]^\beta ln\left[\frac{\theta e^{-\theta x_i}+\Gamma(\alpha)-\theta^\alpha L}{\theta+\Gamma(\alpha)}\right]$$

$$\frac{\partial\ell}{\partial\lambda} = \frac{n}{\lambda} + \sum_{i=1}^n ln\left[1-\left\{\frac{\theta e^{-\theta x_i}+\Gamma(\alpha)-\theta^\alpha L}{\theta+\Gamma(\alpha)}\right\}^\beta\right]$$

$$\frac{\partial\ell}{\partial\theta} = \frac{n}{\theta} - \frac{n}{\theta+\Gamma(\alpha)} - \sum_{i=1}^n x_i + \sum_{i=1}^n\left[\frac{(\alpha-1)\theta^{\alpha-2}(x_i)^{\alpha-1}}{\theta+\theta^{\alpha-1}(x_i)^{\alpha-1}}\right]$$
$$+(\beta-1)\sum_{i=1}^n\left[\frac{(\theta-\theta\alpha-\alpha\Gamma(\alpha))\theta^{\alpha-1}L-(\theta+\Gamma(\alpha))\theta^\alpha L'-(\theta^2x_i+\theta x_i\Gamma(\alpha)+\Gamma(\alpha))e^{-\theta x_i}-\Gamma(\alpha)}{(\theta e^{-\theta x_i}+\Gamma(\alpha)-\theta^\alpha L)(\theta+\Gamma(\alpha))}\right]$$
$$-\beta(\lambda-1)\sum_{i=1}^n\left[\left[1-\left\{\frac{\theta e^{-\theta x_i}+\Gamma(\alpha)-\theta^\alpha L}{\theta+\Gamma(\alpha)}\right\}^\beta\right]^{-1}\left[\frac{\theta e^{-\theta x_i}+\Gamma(\alpha)-\theta^\alpha L}{\theta+\Gamma(\alpha)}\right]^{\beta-1}\left[\frac{(\theta-\theta\alpha-\alpha\Gamma(\alpha))\theta^{\alpha-1}L-(\theta+\Gamma(\alpha))\theta^\alpha L'-(\theta^2x_i+\theta x_i\Gamma(\alpha)+\Gamma(\alpha))e^{-\theta x_i}-\Gamma(\alpha)}{(\theta+\Gamma(\alpha))^2}\right]\right]$$

Where,

$$L = \int_0^x t^{\alpha-1}e^{-\theta t}dt \quad and\ L' = \frac{d}{d\theta}\left(\int_0^x t^{\alpha-1}e^{-\theta t}dt\right)$$

By setting $\frac{\partial\ell}{\partial\alpha} = \frac{\partial\ell}{\partial\beta} = \frac{\partial\ell}{\partial\lambda} = \frac{\partial\ell}{\partial\theta} = 0$ and solving them for α,β,λ and θ, we get the ML estimates of the $EGnEG\ (\alpha,\beta,\lambda\ \theta)$distribution. But normally, it is not possible to solve non-linear equations (32) so with help of suitable R package, one can solve them easily. Let $\underline{\Theta} = (\alpha,\beta,\lambda,\theta)$denote the parameter vector of $EGnEG\ (\alpha,\beta,\lambda\ \theta)$ and the MLE of $\underline{\Theta}\ as\underline{\widehat{\Theta}} = (\hat{\alpha},\hat{\beta},\hat{\lambda},\hat{\theta})$, then the asymptotic normality results in $(\underline{\widehat{\Theta}}-\underline{\Theta}) \to N_4\left[0,\left(I\left(\underline{\Theta}\right)\right)^{-1}\right]$ where $I(\underline{\Theta})$the Fisher information matrix is given by,

$$I(\underline{\Theta}) = \begin{bmatrix} E\left(\frac{\partial^2 l}{\partial\alpha^2}\right) & E\left(\frac{\partial^2 l}{\partial\alpha\partial\beta}\right) & E\left(\frac{\partial^2 l}{\partial\alpha\partial\lambda}\right) & E\left(\frac{\partial^2 l}{\partial\alpha\partial\theta}\right) \\ E\left(\frac{\partial^2 l}{\partial\beta\partial\alpha}\right) & E\left(\frac{\partial^2 l}{\partial\beta^2}\right) & E\left(\frac{\partial^2 l}{\partial\beta\partial\lambda}\right) & E\left(\frac{\partial^2 l}{\partial\beta\partial\theta}\right) \\ E\left(\frac{\partial^2 l}{\partial\lambda\partial\alpha}\right) & E\left(\frac{\partial^2 l}{\partial\lambda\partial\beta}\right) & E\left(\frac{\partial^2 l}{\partial\lambda^2}\right) & E\left(\frac{\partial^2 l}{\partial\lambda\partial\theta}\right) \\ E\left(\frac{\partial^2 l}{\partial\theta\partial\alpha}\right) & E\left(\frac{\partial^2 l}{\partial\theta\partial\beta}\right) & E\left(\frac{\partial^2 l}{\partial\theta\partial\lambda}\right) & E\left(\frac{\partial^2 l}{\partial\theta^2}\right) \end{bmatrix}$$

Hence by plugging in the estimated values of the parameters, we approximate the asymptotic variance. An estimate of the information matrix $I(\underline{\Theta})$ given by the observed Fisher information matrix $O(\underline{\widehat{\Theta}})$ as

$$O(\underline{\hat{\Theta}}) = \begin{bmatrix} \left(\frac{\partial^2 l}{\partial\alpha^2}\right) & \left(\frac{\partial^2 l}{\partial\alpha\partial\beta}\right) & \left(\frac{\partial^2 l}{\partial\alpha\partial\lambda}\right) & \left(\frac{\partial^2 l}{\partial\alpha\partial\theta}\right) \\ \left(\frac{\partial^2 l}{\partial\beta\partial\alpha}\right) & \left(\frac{\partial^2 l}{\partial\beta^2}\right) & \left(\frac{\partial^2 l}{\partial\beta\partial\lambda}\right) & \left(\frac{\partial^2 l}{\partial\beta\partial\theta}\right) \\ \left(\frac{\partial^2 l}{\partial\lambda\partial\alpha}\right) & \left(\frac{\partial^2 l}{\partial\lambda\partial\beta}\right) & \left(\frac{\partial^2 l}{\partial\lambda^2}\right) & \left(\frac{\partial^2 l}{\partial\lambda\partial\theta}\right) \\ \left(\frac{\partial^2 l}{\partial\theta\partial\alpha}\right) & \left(\frac{\partial^2 l}{\partial\theta\partial\beta}\right) & \left(\frac{\partial^2 l}{\partial\theta\partial\lambda}\right) & \left(\frac{\partial^2 l}{\partial\theta^2}\right) \end{bmatrix}_{(\hat{\alpha},\hat{\beta},\hat{\lambda},\hat{\theta})} = -H(\underline{\Theta})_{(\underline{\Theta}-\underline{\hat{\Theta}})}$$

where H is the Hessian matrix.

The Newton-Raphson algorithm is used in this paper to maximize the likelihood estimate which produces the observed information matrix. Therefore, the variance-covariance matrix is given by,

$$\left[-H(\underline{\Theta})_{(\underline{\Theta}-\underline{\hat{\Theta}})}\right]^{-1} = \begin{bmatrix} var(\hat{\alpha}) & cov(\hat{\alpha},\hat{\beta}) & cov(\hat{\alpha},\hat{\lambda}) & cov(\hat{\alpha},\hat{\theta}) \\ cov(\hat{\beta},\hat{\alpha}) & var(\hat{\beta}) & cov(\hat{\beta},\hat{\lambda}) & cov(\hat{\beta},\hat{\theta}) \\ cov(\hat{\lambda},\hat{\alpha}) & cov(\hat{\lambda},\hat{\beta}) & var(\hat{\lambda}) & cov(\hat{\lambda},\hat{\theta}) \\ cov(\hat{\theta},\hat{\alpha}) & cov(\hat{\theta},\hat{\beta}) & cov(\hat{\theta},\hat{\lambda}) & var(\hat{\theta}) \end{bmatrix} \quad (33)$$

Hence from the asymptotic normality of MLEs, approximate $100(1-\alpha)\%$ confidence intervals for α, β, λ and θ can be constructed as,

$$\hat{\alpha} \pm Z_{\alpha/2}\sqrt{var(\hat{\alpha})}, \hat{\beta} \pm Z_{\alpha/2}\sqrt{var(\hat{\beta})}, \hat{\lambda} \pm Z_{\alpha/2}\sqrt{var(\hat{\lambda})},\ \hat{\theta} \pm Z_{\alpha/2}\sqrt{var(\hat{\theta})}$$

Where $Z_{\alpha/2}$ is the upper percentile of standard normal variate.

Method of Least-Square Estimation (LSE)

The LSE of the unknown parameters α, β, λ and θ of $EGnEG$ $(\alpha, \beta, \lambda\ \theta)$distribution can be obtained by using the principle of optimization. Here, we have estimated parameters by minimizing

$$A(x; \alpha, \beta, \lambda, \theta) = \sum_{i=1}^{n}\left[F(X_{(i)}) - \frac{i}{n+1}\right]^2 \quad (34)$$

With respect to unknown parameters $\alpha, \beta, \lambda, and\ \theta$

Suppose $F(X_{(i)})$ denotes the CDF of the ordered random variables $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$ where $\{X_1, X_2, \ldots, X_n\}$ is a random sample of size n taken from a distribution function F(.). The least-square estimators of $\alpha, \beta, \lambda, and\ \theta$ say $\hat{\alpha}, \hat{\beta}, \hat{\lambda},\ and\ \hat{\theta}$ respectively, can be obtained by minimizing.

$$A(x; \alpha, \beta, \lambda, \theta) = \sum_{i=1}^{n}\left[\left[1 - \left\{\frac{[\theta e^{-\theta x} + \Gamma(\alpha) - \theta^\alpha L]}{\theta + \Gamma(\alpha)}\right\}^\beta\right]^\lambda - \frac{i}{n+1}\right]^2 ; x > 0, (\alpha, \beta, \lambda, \theta) > 0 \ldots \quad (35)$$

With respect to unknown parameters $\alpha, \beta, \lambda, and\ \theta$

Differentiating (35) with respect to $\alpha, \beta, \lambda, and\ \theta$ we get

$$\frac{\partial A}{\partial \alpha} = -2\beta\lambda\sum_{I=1}^{n}\left[\begin{matrix}\left[\left[1 - \left\{\frac{[\theta e^{-\theta x} + \Gamma(\alpha) - \theta^\alpha L]}{\theta + \Gamma(\alpha)}\right\}^\beta\right]^\lambda - \frac{i}{n+1}\right]\left[1 - \left\{\frac{\theta e^{-\theta x_i} + \Gamma(\alpha) - \theta^\alpha L}{\theta + \Gamma(\alpha)}\right\}^\beta\right]^{\lambda-1} \\ \times\left\{\frac{\theta e^{-\theta x_i} + \Gamma(\alpha) - \theta^\alpha L}{\theta + \Gamma(\alpha)}\right\}^{\beta-1}\left[\frac{(\theta - \theta e^{-\theta x_i} + \theta^\alpha L)\Gamma'(\alpha) - (\theta + \Gamma(\alpha))\theta^\alpha L ln\theta}{(\theta + \Gamma(\alpha))^2}\right]\end{matrix}\right]$$

$$\frac{\partial A}{\partial \beta} = -2\lambda\sum_{I=1}^{n}\left[\begin{matrix}\left[\left[1 - \left\{\frac{[\theta e^{-\theta x} + \Gamma(\alpha) - \theta^\alpha L]}{\theta + \Gamma(\alpha)}\right\}^\beta\right]^\lambda - \frac{i}{n+1}\right]\left[1 - \left\{\frac{\theta e^{-\theta x_i} + \Gamma(\alpha) - \theta^\alpha L}{\theta + \Gamma(\alpha)}\right\}^\beta\right]^{\lambda-1} \\ \times\left\{\frac{\theta e^{-\theta x_i} + \Gamma(\alpha) - \theta^\alpha L}{\theta + \Gamma(\alpha)}\right\}^{\beta} ln\left[\frac{\theta e^{-\theta x_i} + \Gamma(\alpha) - \theta^\alpha L}{\theta + \Gamma(\alpha)}\right]\end{matrix}\right]$$

$$\frac{\partial A}{\partial \lambda} = -2\sum_{I=1}^{n}\left[\begin{matrix}\left[\left[1 - \left\{\frac{[\theta e^{-\theta x} + \Gamma(\alpha) - \theta^\alpha L]}{\theta + \Gamma(\alpha)}\right\}^\beta\right]^\lambda - \frac{i}{n+1}\right]\left[1 - \left\{\frac{\theta e^{-\theta x_i} + \Gamma(\alpha) - \theta^\alpha L}{\theta + \Gamma(\alpha)}\right\}^\beta\right]^{\lambda} \\ \times ln\left[1 - \left\{\frac{\theta e^{-\theta x_i} + \Gamma(\alpha) - \theta^\alpha L}{\theta + \Gamma(\alpha)}\right\}^\beta\right]\end{matrix}\right]$$

$$\frac{\partial A}{\partial \theta}=-2\beta\lambda\sum_{I=1}^{n}\left[\begin{array}{c}\left[\left[1-\left\{\frac{\left[\theta e^{-\theta x}+\Gamma(\alpha)-\theta^{\alpha}L\right]}{\theta+\Gamma(\alpha)}\right\}^{\beta}\right]^{\lambda}-\frac{i}{n+1}\right]\left[1-\left\{\frac{\theta e^{-\theta x_i}+\Gamma(\alpha)-\theta^{\alpha}L}{\theta+\Gamma(\alpha)}\right\}^{\beta}\right]^{\lambda-1}\\ \left\{\frac{\theta e^{-\theta x_i}+\Gamma(\alpha)-\theta^{\alpha}L}{\theta+\Gamma(\alpha)}\right\}^{\beta-1}\\ \left[\frac{\left(\theta-\theta\alpha-\alpha\Gamma(\alpha)\right)\theta^{\alpha-1}L-\left(\theta+\Gamma(\alpha)\right)\theta^{\alpha}L'-\left(\theta^{2}x_i+\theta x_i\Gamma(\alpha)+\Gamma(\alpha)\right)e^{-\theta x_i}-\Gamma(\alpha)}{\left(\theta+\Gamma(\alpha)\right)^{2}}\right]\end{array}\right]$$

Where, $L=\int_0^x t^{\alpha-1}e^{-\theta t}dt \quad and\ L'=\frac{d}{d\theta}\left(\int_0^x t^{\alpha-1}e^{-\theta t}dt\right)$

In similar manner, we can estimate the weighted least square estimators by minimizing

$$D(x;\alpha,\beta,\lambda,\theta)=\sum_{i=1}^{n}w_i\left[F\left(X_{(i)}\right)-\frac{i}{n+1}\right]^{2}=\sum_{i=1}^{n}w_i\left[\left[1-\left\{\frac{\left[\theta e^{-\theta x}+\Gamma(\alpha)-\theta^{\alpha}L\right]}{\theta+\Gamma(\alpha)}\right\}^{\beta}\right]^{\lambda}-\frac{i}{n+1}\right]^{2}$$

With respect to $\alpha,\beta,\lambda, and\ \theta$. The weights w_i are computed as $w_i=\frac{1}{Var\left(X_{(i)}\right)}=\frac{(n+1)^2(n+2)}{i(n-i+1)}$

Hence, the weighted least square estimators of $\alpha,\beta,\lambda, and\ \theta$ respectively can be obtained by minimizing,

$$D(X;\alpha,\beta,\lambda,\theta)=\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[\left[1-\left\{\frac{\left[\theta e^{-\theta x}+\Gamma(\alpha)-\theta^{\alpha}L\right]}{\theta+\Gamma(\alpha)}\right\}^{\beta}\right]^{\lambda}-\frac{i}{n+1}\right]^{2} \tag{36}$$

Method of Cramer-Von-Mises estimation (CVME)

The Cramer-Von-Mises-estimators of $\alpha,\beta,\lambda, and\ \theta$ are obtained by minimizing the function

$$Z(X;\alpha,\beta,\lambda,\theta)=\frac{1}{12n}+\sum_{i=1}^{n}\left[F\left(\frac{X_{i:n}}{\alpha},\beta,\lambda,\theta\right)-\frac{2i-1}{2n}\right]^{2}$$

$$Z(X;\alpha,\beta,\lambda,\theta)=\frac{1}{12n}+\sum_{i=1}^{n}\left[\left[1-\left\{\frac{\left[\theta e^{-\theta x}+\Gamma(\alpha)-\theta^{\alpha}L\right]}{\theta+\Gamma(\alpha)}\right\}^{\beta}\right]^{\lambda}-\frac{2i-1}{2n}\right]^{2} \tag{37}$$

Differentiating (31) with respect to $\alpha,\beta,\lambda, and\ \theta$ we get

$$\frac{\partial Z}{\partial \alpha}=-2\beta\lambda\sum_{I=1}^{n}\left[\begin{array}{c}\left[\left[1-\left\{\frac{\left[\theta e^{-\theta x}+\Gamma(\alpha)-\theta^{\alpha}L\right]}{\theta+\Gamma(\alpha)}\right\}^{\beta}\right]^{\lambda}-\frac{2i-1}{2n}\right]\left[1-\left\{\frac{\theta e^{-\theta x_i}+\Gamma(\alpha)-\theta^{\alpha}L}{\theta+\Gamma(\alpha)}\right\}^{\beta}\right]^{\lambda-1}\\ \left\{\frac{\theta e^{-\theta x_i}+\Gamma(\alpha)-\theta^{\alpha}L}{\theta+\Gamma(\alpha)}\right\}^{\beta-1}\left[\frac{\left(\theta-\theta e^{-\theta x_i}+\theta^{\alpha}L\right)\Gamma'(\alpha)-\left(\theta+\Gamma(\alpha)\right)\theta^{\alpha}L\ln\theta}{\left(\theta+\Gamma(\alpha)\right)^{2}}\right]\end{array}\right] \tag{38}$$

$$\frac{\partial Z}{\partial \beta}=-2\lambda\sum_{I=1}^{n}\left[\begin{array}{c}\left[\left[1-\left\{\frac{\left[\theta e^{-\theta x}+\Gamma(\alpha)-\theta^{\alpha}L\right]}{\theta+\Gamma(\alpha)}\right\}^{\beta}\right]^{\lambda}-\frac{2i-1}{2n}\right]\left[1-\left\{\frac{\theta e^{-\theta x_i}+\Gamma(\alpha)-\theta^{\alpha}L}{\theta+\Gamma(\alpha)}\right\}^{\beta}\right]^{\lambda-1}\\ \left\{\frac{\theta e^{-\theta x_i}+\Gamma(\alpha)-\theta^{\alpha}L}{\theta+\Gamma(\alpha)}\right\}^{\beta}\ ln\left[\frac{\theta e^{-\theta x_i}+\Gamma(\alpha)-\theta^{\alpha}L}{\theta+\Gamma(\alpha)}\right]\end{array}\right] \tag{39}$$

$$\frac{\partial Z}{\partial \lambda}=-2\sum_{I=1}^{n}\left[\begin{array}{c}\left[\left[1-\left\{\frac{\left[\theta e^{-\theta x}+\Gamma(\alpha)-\theta^{\alpha}L\right]}{\theta+\Gamma(\alpha)}\right\}^{\beta}\right]^{\lambda}-\frac{2i-1}{2n}\right]\left[1-\left\{\frac{\theta e^{-\theta x_i}+\Gamma(\alpha)-\theta^{\alpha}L}{\theta+\Gamma(\alpha)}\right\}^{\beta}\right]^{\lambda}\\ ln\left[1-\left\{\frac{\theta e^{-\theta x_i}+\Gamma(\alpha)-\theta^{\alpha}L}{\theta+\Gamma(\alpha)}\right\}^{\beta}\right]\end{array}\right] \tag{40}$$

$$\frac{\partial Z}{\partial \theta}=-2\beta\lambda\sum_{I=1}^{n}\left[\begin{array}{c}\left[\left[1-\left\{\frac{\left[\theta e^{-\theta x}+\Gamma(\alpha)-\theta^{\alpha}L\right]}{\theta+\Gamma(\alpha)}\right\}^{\beta}\right]^{\lambda}-\frac{2i-1}{2n}\right]\left[1-\left\{\frac{\theta e^{-\theta x_i}+\Gamma(\alpha)-\theta^{\alpha}L}{\theta+\Gamma(\alpha)}\right\}^{\beta}\right]^{\lambda-1}\\ \left\{\frac{\theta e^{-\theta x_i}+\Gamma(\alpha)-\theta^{\alpha}L}{\theta+\Gamma(\alpha)}\right\}^{\beta-1}\\ \left[\frac{\left(\theta-\theta\alpha-\alpha\Gamma(\alpha)\right)\theta^{\alpha-1}L-\left(\theta+\Gamma(\alpha)\right)\theta^{\alpha}L'-\left(\theta^{2}x_i+\theta x_i\Gamma(\alpha)+\Gamma(\alpha)\right)e^{-\theta x_i}-\Gamma(\alpha)}{\left(\theta+\Gamma(\alpha)\right)^{2}}\right]\end{array}\right] \tag{41}$$

Where, $L=\int_0^x t^{\alpha-1}e^{-\theta t}dt \quad and\ L'=\frac{d}{d\theta}\left(\int_0^x t^{\alpha-1}e^{-\theta t}dt\right)$

By setting $\frac{\partial Z}{\partial \alpha} = 0, \frac{\partial Z}{\partial \beta} = 0, \frac{\partial Z}{\partial \lambda} = 0 \; and \; \frac{\partial Z}{\partial \theta} = 0$ simultaneously, we obtained the CVM estimators.

The maximum likelihood estimates of the parameters α, β, λ and θ respectively are taken by solving the equations (38 to 41). But they cannot be solved analytically because they are not expressed in closed form. These equations can be solved using any numerical method such as the Newton-Raphson method (Henningsen & Toomet, 2011), the Nelder-Mead method (Nelder &Mead, 1965), BFGS method (Fletcher, 1987), SANN method (Belisle, 1992), and the like. The Newton-Raphson method is however employed in this paper with the use of the optim function in R package (R Core Team, 2018) and Henningsen & Toomet, (2011) to solve the equations iteratively.

Applications

In this section, the goodness-of-fit of the distribution is discussed with an application to real-life datasets. The parameters of the distribution were solved using the MLE method while the goodness-of-fit was evaluated using the Akaike Information Criterion (AIC), Akaike Information Criterion Corrected (AICC), Bayesian Information Criterion (BIC) and –2logLik with their respective statistics given below.

$$AIC = -2\ln L + 2k \qquad (42)$$

$$BIC = -2\ln L + k\ln n \qquad (43)$$

where k is the number of parameters and n is the sample size. The distribution that has a lower value of these criteria is judged to be the best among others.

Data Description: This represents an uncensored dataset corresponding to Remission times (in months) of a random sample of 118 Bladder cancer patients reported in Lee & Wang (2003) and used by Umar et al (2019) to mention but few.

Table 1: Summary Statistics of the dataset used (Remission times (in months) of Bladder cancer data)

Statistic	Mean	Median	Min	Max	1st Qu.	3rd Qu.	Std Dev	Skew	Kurtosis
Value	7.546	5.34	0.08	79.05	2.535	11.4	8.94	1.2	3.5

The mean (7.546) and median (5.340) provide different perspectives of central tendency. The mean is higher than the median, suggesting a right-skewed distribution where a few large values pull the mean upwards. The standard deviation (8.94) is large, indicating that the data points are widely spread out from the mean. The positive skew (1.20) tells us that the data has a rightward tail with some extreme high values, which is supported by the large maximum value (79.05).The kurtosis (3.50) indicates that the distribution has heavy tails, which could mean that there are a few extreme values (outliers) in the dataset. Therefore, the data appears to be right-skewed with a few large values that are influencing the mean and other statistics. The dataset has a relatively wide range, and the presence of heavy tails suggests that outliers may play an important role in the overall distribution of the data.

Table 1: Parameters Estimation and Goodness-of-Fit Test Results of the EGnEG and other Existing distributions for the Remission times (in months) of Bladder cancer data

DISTRIBUTION	**ML ESTIMATE**	**-2logLik**	**AIC**	**BIC**
EGnEG	$\hat{\alpha} = 1.041$	699.917	707.917	708.677
	$\hat{\beta} = 3.002$			
	$\hat{\lambda} = 0.007$			
	$\hat{\theta} = 0.086$			
ENEGD	$\hat{\alpha} = 0.961$	711.091	717.091	717.862
	$\hat{\theta} = 0.141$			
	$\hat{\lambda} = 3.071$			
NEGD	$\hat{\alpha} = 1.462$	742.679	746.679	747.449
	$\hat{\theta} = 0.104$			
EGD	$\hat{\alpha} = 1.946$	731.254	735.254	738.502
	$\hat{\theta} = 0.002$			
GD	$\hat{\alpha} = 0.386$	742.234	746.234	747.005

	$\hat{\theta} = 0.744$			
ELD	$\hat{\alpha} = 0.791$	739.584	745.368	744.139
	$\hat{\theta} = 0.557$			
LD	$\hat{\theta} = 0.196$	839.060	841.060	843.912
EED	$\hat{\alpha} = 0.930$	726.695	730.695	745.778
	$\hat{\theta} = 0.111$			
ED	$\hat{\theta} = 0.116$	743.718	745.718	748.489

RESULTS AND DISCUSSION

The analysis of the remission times for Bladder cancer data using various lifetime distributions revealed that the Exponentiated Generalized New Exponential-Gamma (EGnEG) distribution provided the best fit compared to other established models. In Table 1, the maximum likelihood (ML) estimates for the parameters of each distribution, along with the corresponding values for -2logLik, AIC, and BIC, are presented for comparison.

The EGnEG distribution, with its four parameters achieved the lowest -2logLik (699.917), AIC (707.917), and BIC (708.677) values, indicating its superior fit to the Bladder cancer remission time data. The low values of AIC and BIC suggest that the EGnEG distribution is not only a good fit but also the most parsimonious model among the tested distributions, which is crucial when balancing model fit and complexity.

The ENEGD presented -2logLik = 711.091, AIC = 717.091, and BIC = 717.862. Although it performed reasonably well, it did not outperform the EGnEG model. The higher AIC and BIC values for ENEGD suggest that the more complex structure of the EGnEG distribution, which includes the additional flexibility provided by the β and λ parameters, leads to a better model fit for this dataset.

The other distributions, such as the New Exponential-Gamma Distribution (NEGD), Exponentiated Generalized Distribution (EGD), Generalized Distribution (GD), Exponentiated Lindley Distribution (ELD), Lindley Distribution (LD), Exponentiated Exponential Distribution (EED), and Exponential Distribution (ED), all exhibited relatively higher AIC and BIC values compared to the EGnEG distribution, suggesting that they were not as well-suited for modeling the remission times of Bladder cancer. Specifically, the NEGD and EGD had -2logLik values of 742.679 and 731.254, respectively, with corresponding AIC and BIC values significantly higher than those of the EGnEG model.

In terms of parameter estimates, the EGnEG distribution's parameter values (e.g., $\theta = 0.086$) indicate a relatively slow rate of remission, which aligns with expectations in the context of medical data, where remission times for cancer patients often exhibit slower decay over time. The shape parameters α and β, as well as the scale parameter θ, provide important insights into the distribution of remission times, allowing for a more nuanced understanding of the remission process.

The superior performance of the EGnEG distribution suggests that it is a robust and flexible model for analyzing lifetime data in medical and clinical contexts, particularly for complex datasets like cancer remission times. The model's ability to capture both the shape of the distribution and the varying hazard rate over time makes it an ideal candidate for modeling survival data with non-constant hazard functions.

The AIC and BIC results also underscore the importance of considering both model fit and parsimony when selecting the best distribution for data modeling. The EGnEG distribution's balance between these factors makes it a highly practical choice for researchers and practitioners in fields such as oncology, where accurate modeling of survival and remission times is essential for decision-making.

Although the EGnEG distribution provided a superior fit to the data in this study, there are still several opportunities for improvement and exploration. For example, further research could investigate the sensitivity of the EGnEG model to different initial parameter estimates or data sets with varying sample sizes. Additionally, while the EGnEG distribution outperformed other models in this study, it would be valuable to explore its application to other types of medical and clinical data, such as mortality rates or recovery times, to confirm its robustness and generalizability.

Moreover, future work could focus on extending the EGnEG model by incorporating covariates that influence the remission time, such as age, treatment type, or genetic factors. This would allow for a more comprehensive understanding of the factors influencing remission and could provide insights into personalized treatment strategies.

Another avenue for future research is the development of computational tools and algorithms that can efficiently estimate the parameters of the EGnEG distribution in large-scale clinical trials, which often involve large datasets with many covariates. By making these tools available, researchers could more easily apply the EGnEG distribution to a wider array of datasets, improving the accuracy of their lifetime data models.

CONCLUSION

An Exponentiated Generalized New Exponential-Gamma (EGnEG) distribution was derived and studied in this paper. The distribution is quite flexible and can model data with various types of behavior, such asHeavy-tailed, Skewed and/or data with different levels of variability depending on the parameters.This distribution is useful in fields like; Reliability analysis, Survival analysis, Queueing models and Financial modeling.
The results of this study demonstrate that the Exponentiated Generalized New Exponential-Gamma (EGnEG) distribution offers a powerful and flexible tool for modeling lifetime data, particularly in the context of medical data such as cancer remission times. Its superior fit, as evidenced by the lower AIC and BIC values compared to other models, suggests that the EGnEG distribution could become a valuable addition to the toolkit for survival analysis and reliability modeling.
Future research should continue to explore the applications and potential extensions of the EGnEG distribution, especially in clinical settings where personalized treatment decisions are becoming increasingly important.

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