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On Substructures and Root Sets in Antimultigroups and Their Direct Products

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ABSTRACT
This paper presents an extension of the direct product operation to antimultigroups. We prove that the direct product of two antimultigroups is itself an antimultigroup, preserving the defining axioms under Cartesian pairing. We introduce and analyze the main substructures of antimultigroups. These substructures include strong and weak upper and lower cuts, and show that each type of cut forms a sub-antimultigroup. Also, we examine the behavior of root sets and the structural connections between cuts under union and intersection. This leads to the establishment that such operations yield sub-antimultigroups under suitable conditions. These findings contribute to a deeper understanding of the structure of antimultigroups. Thus, it lays the groundwork for further developments in antimultigroup theory.

Keywords:
multisets,
multigroup,
group,
antimultigroup,
direct product.

INTRODUCTION

We begin by reviewing some established definitions and results in the realm of multigroup theory. These foundational concepts serve as a basis for the development of new ideas in this work. Additionally, we introduce novel definitions and results that are integral to the discussions and analyses presented. The first key concept that we revisit is multiset and their significance in generalizing the traditional set theory. Multisets, as introduced by N. G. De Bruijn (DeBruijn 1983), allow for the repetition of elements within an unordered collection, thus expanding the scope beyond the constraints of Cantor's crisp set. Understanding the properties and structures of multisets is essential in exploring the extensions of traditional group theory into the realm of multigroups.

The field of group theory, rooted in George Cantor's set theory principles, has long been a significant area of mathematical study. Cantor's original set theory, which prohibited element repetition, laid the foundation for group theory (Kleiner, 1986). Over time, as mathematical research progressed, it became apparent that this limitation needed to be addressed. The introduction of multisets, proposed by N. G. De Bruijn to Knuth, provided a solution by allowing for the repetition of elements within an unordered collection (Knuth, 1981). Multisets have found wide-ranging applications in various fields such as database systems, biological systems, and information retrieval (Blizard, 1991; Singh 1994; Singh *et al.*, 2007 & 2008).

Furthermore, we delve into the definition and properties of multigroups, which are algebraic systems that adhere to group theory axioms. The evolution of multigroup theory, incorporating the principles of multisets and other non-classical groups, has paved the way for a greater understanding of algebraic structures. By synthesizing existing knowledge with new contributions, we comprehensively examine the direct product of antimultigroups and lay the groundwork for further exploration and analysis in subsequent sections of this paper (Ejegwa and Ibrahim, 2017; Ejegwa, 2020).

Dresher and Ore (1938) introduced multigroups as algebraic systems satisfying group theory axioms with multivalued multiplication. However, this definition did not align with the properties of multisets or other non-classical groups like fuzzy groups, soft groups, and intuitionistic fuzzy groups as seen in Rosenfeld (1971), Aktas and Cagman, (2007), Biswas, (1989) Nazmul and Samanta, (2011 and 2015), Shinoj *et al.*, (2015) and Shinoj and Sunil, (2015). Subsequent research efforts aimed at refining the concept of multigroup by integrating it with multisets, leading to a more coherent and comprehensive definition. The notion of the direct product of antimultigroups, an extension of multigroup theory in reverse order, and its properties are presented. See Peter *et al.*, (2024), Peter and Abdullahi (2025a), Peter and Abdullahi (2025b), Peter, (2025) are some texts with further contributions in the parlance of multigroupand antimultigroup – the last three of which are based on Singh's dressed epsilon notations.

MATERIALS AND METHODS

Preliminaries

In this section, we review some existing definitions and results. We also introduce new definitions and results that will be used in this work.

Definition 1 (Singh *et al.*, 2007**)**
Let $X = \{x_1, x_2, \dots, x_j, \dots\}$ be a set. A *multiset* A over X is a function:
$A: X \to \mathbb{N} = \{0,1,2,\dots\}$
such that for each $x \in \mathrm{Dom}(A), A(x) = m_A(x) > 0$, where $m_A(x)$ denotes the *multiplicity* of the elements x in A. The set of all multisets over X is denoted by $\mathcal{M}(X)$.

Definition 2 (Syropoulos, 2011)
Let A and B be multisets. We say that A is a *submultiset* (or *multisubset*) of B, written $A \subseteq B$ (or $B \supseteq A$), if

$$m_A(x) \leq m_B(x) \text{for all } x \in D$$

where D is the root set of B. If $A \subseteq B$ and $A \neq B$, then A is called a *proper submultiset* of B.

Definition 3 (Namzul *et al.*, 2013)
Let X be a group. A multiset A over X is said to be a multigroup over X if its multiplicity function m_A satisfies the following two conditions for all $x, y \in X$:

1. $m_A(xy) \geq \min\{m_A(x), m_A(y)\}$
2. $m_A(x^{-1}) \geq m_A(x)$.

It follows immediately from (2) that

$$m_A(x^{-1}) = m_A(x),$$

Since $m_A(x) = m_A(x^{-1})^{-1} \geq m_A(x^{-1})$ and vice versa. The set of all multigroups over X is denoted by $\mathcal{MG}(X)$. Thus, A is a multigroup over X if and only if the multiplicity function respects the group multiplication and inversion as stated above.

Definition 4 (Ejegwa, 2020)
Let X be a groupoid. A multiset A over X is called an *antimultigroupoid* of X if

$$m_A(xy) \leq m_A(x) \vee m_A(y) \text{for all } x, y \in X,$$

where $\vee$ denotes the maximum of the two values.

Definition 5 (Ejegwa, 2020)
A multiset A over a group X is said to be an *antimultigroup* if the following conditions hold:

1. $m_A(xy) \leq m_A(x) \vee m_A(y)$ for all $x, y \in X$,
2. $m_A(x^{-1}) \leq m_A(x)$ for all $x \in X$

The set of all antimultigroups over X is denoted by $\mathcal{AMG}(X)$.

Example 1
Let $X = \{e, a, b, c\}$ be the Klein four-group, with operation defined by
$ab = c, ac = b, bc = a, a^2 = b^2 = c^2 = e$.
Then the multiset

$$A = \{e^2, a^5, b^4, c^5\}$$

Is an antimultigroup over X.

Proposition 1 (Ejegwa, 2020)
Let $A \in \mathcal{AMG}(X)$, where e is the identity elements of X. Then the following conditions hold:

1. $m_A(e) \leq m_A(x)$ *for all* $x \in X$,
2. $m_A(x^n) \leq m_A(x)$ *for all* $x \in X, n \in \mathbb{N}$,
3. $m_A(x^{-1}) \leq m_A(x)$ *for all* $x \in X$

Definition 6 (Nazmul *et al.*, 2013)
Let $A \in \mathcal{M}(X)$ be a multiset over X. Define the following subsets:

1. $A_* = \{x \in X \mid m_A(x) > 0$, called the support of A
2. $A_* = \{x \in X \mid m_A(x) = m_A(e)$, where e is the idendity element of X.

Definition 7 (Ejegwa, 2020)
Let $A \in \mathcal{AMG}(X)$. For every $n \in \mathbb{N}$, define the cut of A at level n as:

$$A_{[n]} = \{x \in X \mid m_A(x) \leq n\}.$$

Proposition 2 (Ejegwa, 2020)
Let $A \in \mathcal{AMG}(X)$. Then for every $n \geq m_A(e)$, the cut $A_{[n]}$ is a sub-antimultigroup of X.

Proposition 3 (Ejegwa, 2020)
Let $A \in \mathcal{AMG}(X)$. A submultset $B \subseteq A$ is called a sub-antimultigroup of A, denoted by $B \leq A$ if $B \in \mathcal{AMG}(X)$; that is, if B forms an antimultigroup under the same binary operation.

If $B \leq A$ and $B \neq A$, then B is called a *proper sub-antimultigroup* of A, denoted by $B < A$.

Example 2
Let $X = \{e, a, b, c\}$ be the Klein four-group (with $ab = c, ac = b, bc = a, a^2 = b^2 = c^2 = e$), and let
$A = \{e^5, a^7, b^6, c^7\}$

be an antimultigroup over X. Then the multisets

$$B = \{e^4, a^6, b^5, c^6\}, C = \{e^3, a^5, b^4, c^5\}$$

are sub-antimultigroups of A.
Since both B and C are strictly contained in A, they are also proper sub-antimultigroups of A; that is, $B < A$ and $C < A$

Definition 8 Let $A \in \mathcal{AMG}(X)$ and $n \in \mathbb{N}$. The following subsets are defined as follows:

1. The *strong upper cut* of A at level n is:

$$A_{[n]} = \{x \in X \mid m_A(x) < n\}.$$

2. The *weak upper cut* of A at level n is:

$$A_{(n)} = \{x \in X \mid m_A(x) \leq n\}.$$

3. The *strong lower cut* of A at level n is:

$$A^{[n]} = \{x \in X \mid m_A(x) \geq n\}.$$

4. The *weak lower cut* of A at level n is:

$$A^{(n)} = \{x \in X \mid m_A(x) > n\}.$$

Proposition 4 (Sub-antimultigroup Property of Cuts);
Let $A \in \mathcal{AMG}(X)$ and $n \in \mathbb{N}$. Then $A_{[n]}$, $A_{(n)}$, $A^{[n]}$ and $A^{(n)}$ are *sub-antimultigroups of* A.

Proof.
Let C be any of the four cuts above with induced multiplicities from A: that is,

$$m_C(x) = \begin{cases} m_A(x), & \text{if } x \in C \\ 0, & \text{otherwise} \end{cases}$$

We now verify that $C \in AMG(X)$:

1. Inverse symmetry:
For all $x \in X$:
Since $A \in AMG(X)$, we have $m_A(x^{-1}) = m_A(x)$. Then for each $C, x \in m_A(x)$ satisfies some inequality involving $n \Leftrightarrow m_A(x^{-1})$ satisfies the same. Hence, $x \in C \Leftrightarrow x^{-1} \in C$, so $m_C(x^{-1}) = m_C(x)$

2. Antimultiplicative inequality:
For all $x, y \in X$:
$m_C(xy) \leq m_C(x) \vee m_C(y)$.

This holds because $m_C(z) \leq m_A(z)$ for all z, and A satisfies the antimultigroup inequality:

$m_A(xy) \leq m_A(x) \vee m_A(y)$.
So if $xy \in C$, then:

$m_C(xy) = m_A(xy) \leq m_A(x) \vee m_A(y)$.

But since $m_C(x) \leq m_A(x), m_C(y) \leq m_A(y)$, the right-hand side is at least $m_C(x) \vee m_C(y)$, so the inequality holds.

Proposition 5 (Ibrahim and Awolola, 2023)
Let $A \in \mathcal{AMG}(X)$. *Then the sets* $A^* = \{x \in X \mid m_A(x) = m_A(e)\}$ *and* $A_* = \{x \in X \mid m_A(x) > 0\}$.

Proposition 6. *Let* $A, B \in \mathcal{AMG}(X)$ *with* $A \subseteq B$. *Assume that the normality condition is given in the stronger form:*
$m_A(xyx^{-1}) = m_A(y)\ \forall x, y \in X$.
Then the following statements are equivalent:

i. *A is a normal sub-antimultigroup of* B.
ii. $m_A(xyx^{-1}) = m_A(y)\ \forall x, y \in X$.
iii. $m_A(xy) = m_A(yx)\ \forall x, y \in X$.

Proof.
(i) ⇒ (ii)
This follows directly from the assumption that normality is defined by equality, i,e,

$m_A(xyx^{-1}) = m_A(y)\ \forall x, y \in X$.
Hence, (i) implies (ii).

(ii) ⇒ (iii)
Assume
$m_A(xyx^{-1}) = m_A(y)\ \forall x, y \in X$.

Let $x, y \in X$. Multiply both sides on the right by x, so

$xyx^{-1}x = xy$.

On the other hand, let $yx = x^{-1}xyx$. Since $A \subseteq B \in \mathcal{AMG}(X)$, the multiplicities are preserved under such transformations (this is assumed or can be derived in an associative multigroup). Hence, using the equality condition repeatedly yields:

$m_A(xy) = m_A(yx)$

Thus, (ii) implies (iii).

(iii) ⇒ (i)
Assume

$m_A(xy) = m_A(yx)\ \forall x, y \in X$

and that $A \subseteq B$, We aim to show that

$m_A(xyx^{-1}) = m_A(y)\ \forall x, y \in X$

Let $x, y \in X$. Then:

$m_A(xyx^{-1}) = m_A(yx^{-1}x) = m_A(y)$,

Since $x^{-1}x = e$, the identity, and multigroup associativity and identity assumptions give

$yx^{-1}x = y$ and $m_A(yx^{-1}x) = m_A(y)$.

Hence, (iii) implies (ii), and thus all three statements are equivalent under the equality assumption.

Direct Product of Antimultigroup

In this section, we introduce the concept of direct product in antimultigroup context and we investigate the properties of direct product of two antimultigroups. Also, we establish some important results with respect to root sets and cuts of antimultigroup.

Definition 9. Let X and Y be groups, $A_1 \in AMG(X)$ and $A_2 \in AMG(Y)$. The *direct product* $A_1 \times A_2$ is a multiset over $X \times Y$ defined by the function:

$$m_{A_1\times A_2}\big((x,y)\big) = m_{A_1}(x) \vee m_{A_2}(y)\ \forall x \in X, \forall y \in Y,$$

where $\vee$ denotes the maximum operator.

Example 3

Let $X = \{e, a,\}$ with $a^2 = e$, and let $Y = \{e', b, c, d\}$ be a Klein 4-group where $b^2 = c^2 = d^2 = e'$). Define:

$$A_1 = \{e^1, a^4\}, \qquad A_2 = \{(e')^2, b^5, c^4, d^5\}.$$

Then $A_1 \in AMG(X)$, $A_2 \in AMG(Y)$, and the product group is:

$$X \times Y = \{(e,e'), (e,b), (e,c), (e,d), (a,e'), (a,b), (a,c), (a,d)\}$$

with identity element (e, e').

From Definition 9, the direct product is:

$$A_1 \times A_2 = \left\{\begin{matrix}(e,e')^2, (e,b)^5, (e,c)^4, (e,d)^5, (a,e')^4, \\ (a,b)^5, (a,c)^4, (a,d)^5\end{matrix}\right\}.$$

This is an antimultigroup over $X \times Y$, satisfying the antimultigroup conditions from Definition 5.

Remark 1.

Note that the total multiplicity of $A_1 \times A_2$ satisfies:

$$|A_1 \times A_2| = \sum_{(x,y)\in X\times Y} m_{A_1\times A_2}\big((x,y)\big) < \left(\sum_{x\in X} m_{A_1}(x)\right).\left(\sum_{y\in Y} m_{A_2}(y)\right) = |A_1|.|A_2|$$

This is different from the classical group case, where the cardinality of the direct product is the product of the cardinalities.

Proposition 7: *Let $A_1 \in AMG(X)$ and $A_2 \in AMG(Y)$ and $A_1 \times A_2 \in MG(X \times Y)$. Then for all $(x,y) \in X \times Y$ the following hold:*

i. $m_{A_1\times A_2}\big((x^{-1}, y^{-1})\big) = m_{A_1\times A_2}\big((x,y)\big)$

ii. $m_{A_1\times A_2}\big((e,e')\big) \le m_{A_1\times A_2}\big((x,y)\big)$

iii. $m_{A_1\times A_2}((x,y)^n) \le m_{A_1\times A_2}\big((x,y)\big)$, *for all* $n \in \mathbb{N}$.

Proof:

Let $(x,y) \in X \times Y$.

(i) $m_{A_1\times A_2}\big((x^{-1}, y^{-1})\big) = m_{A_1}(x^{-1}) \vee m_{A_2}(y^{-1})$
$= m_{A_1}(x) \vee m_{A_2}(y^{-1})$ (by antimultigroup property)
$= m_{A_1\times A_2}\big((x,y)\big)$

(ii) $m_{A_1\times A_2}\big((e,e')\big) = m_{A_1\times A_2}\big((x,y)(x^{-1},y^{-1})\big)$
$\le m_{A_1\times A_2}\big((x,y)\big) \vee m_{A_1\times A_2}\big((x^{-1},y^{-1})\big)$
$= m_{A_1\times A_2}\big((x,y)\big) \vee m_{A_1\times A_2}\big((x,y)\big)$
$= m_{A_1\times A_2}\big((x,y)\big)$

(iii) We prove by induction that for all $n \in \mathbb{N}$,
$m_{A_1\times A_2}((x,y)^n) = m_{A_1\times A_2}\big((x^n, y^n)\big)$
$\le m_{A_1\times A_2}\big((x,y)\big)$.

Base case $n = 1$:
$m_{A_1\times A_2}((x,y)^1) = m_{A_1\times A_2}\big((x^n, y^n)\big)$
$\le m_{A_1\times A_2}\big((x,y)\big)$.

Inductive steps: Suppose
$m_{A_1\times A_2}((x,y)^k) \le m_{A_1\times A_2}\big((x,y)\big)$ for some $k \in \mathbb{N}$.
We want to show that
$m_{A_1\times A_2}((x,y)^{k+1}) \le m_{A_1\times A_2}\big((x,y)\big)$.
Now: $m_{A_1\times A_2}((x,y)^{k+1}) = m_{A_1\times A_2}((x,y)^k(x,y))$
$\le m_{A_1\times A_2}((x,y)^k) \vee m_{A_1\times A_2}((x,y))$
$\le m_{A_1\times A_2}\big((x,y)\big) \vee m_{A_1\times A_2}((x,y))$
$\le m_{A_1\times A_2}\big((x,y)\big)$.

Proposition 8: *Let $A_1, B_1 \in AMG(X)$ and $A_2, B_2 \in AMG(Y)$ and $m, n \in \mathbb{N}$. Then:*

i. $(A_1 \times A_2)_{[n]} \subseteq (A_1 \times A_2)_{[m]}$ *if and only if* $n \le m$,

ii. $A_1 \times A_2 \subseteq B_1 \times B_2$ *if and only if* $(A_1 \times A_2)_{[n]} \subseteq (B_1 \times B_2)_{[n]}$.

Proof.

(i) Assume $(x,y) \in (A_1 \times A_2)_{[n]}$. Then

$$m_{A_1\times A_2}\big((x,y)\big) \le n.$$

Since $n \le m$, it follows that:

$$m_{A_1\times A_2}\big((x,y)\big) \le m$$

Hence, $(A_1 \times A_2)_{[n]} \subseteq (A_1 \times A_2)_{[m]}$.

Conversely, if $(A_1 \times A_2)_{[n]} \subseteq (A_1 \times A_2)_{[m]}$ it is clear that $n \le m$.

(ii) Suppose $A_1 \times A_2 \subseteq B_1 \times B_2$. Then:

$$m_{A_1\times A_2}\big((x,y)\big) \le m_{B_1\times B_2}\big((x,y)\big)\ \forall (x,y)$$

So if $(x,y) \in (A_1\times A_2)_{[n]}$, then:

$$m_{A_1\times A_2}\big((x,y)\big) \le n \Longrightarrow m_{B_1\times B_2}\big((x,y)\big) \le n.$$

Thus, $(x,y) \in (B_1\times B_2)_{[B]}$.

The converse is straightforward.
Since
$A_1\times A_2 \subseteq B_1\times B_2 \Rightarrow m_{A_1\times A_2}\big((x,y)\big) \le m_{B_1\times B_2}\big((x,y)\big)\ \forall (x,y)\in X\times Y$.
For $(x,y)\in (A_1\times A_2)_{[n]}$ and $(x,y)\in (B_1\times B_2)_{[n]}$
$\Rightarrow m_{A_1\times A_2}(x,y) \le m_{B_1\times B_2}(x,y) \le n$.
So, $A_1\times A_2 \subseteq B_1\times B_2$.
The converse is straightforward.

Remark 2: Let $A_1, B_1 \in AMG(X)$ and $A_2, B_2 \in AMG(Y)$ and $m, n \in \mathbb{N}$. Then:

i. $(A_1\times A_2)^{[n]} \subseteq (A_1\times A_2)^{[m]}$ if and only if $n \ge m$,
ii. $A_1\times A_2 \subseteq B_1\times B_2$ if and only if $(A_1\times A_2)_{[n]} \subseteq (B_1\times B_2)_{[n]}$.

Theorem 1: *Let $A_1 \in AMG(X)$ and $A_2 \in AMG(Y)$ respectively. Then for all $n\in\mathbb{N}$,*

$$(A_1\times A_2)_{[n]} = A_{1_{[n]}} \times A_{2_{[n]}}.$$

.

Proof.
Let $(x,y)\in (A_1\times A_2)_{[n]}$. From definition 9 we have,

$$m_{A_1\times A_2}\big((x,y)\big) = \Big(m_{A_1}(x) \vee m_{A_2}(y)\Big) \le n.$$

This implies that $m_{A_1}(x) \le n$ and $m_{A_2}(y) \le n$, hence $x \in A_{1_{[n]}}$ and $y \in A_{2_{[n]}}$, so

$$(x,y) \in A_{1_{[n]}} \times A_{2_{[n]}}$$

Conversely, let $(x,y)\in A_{1_{[n]}} \times A_{2_{[n]}}$, i.e, $m_{A_1}(x) \le n$ and $m_{A_2}(y) \le n$. Then

$$m_{A_1\times A_2}\big((x,y)\big) = \Big(m_{A_1}(x) \vee m_{A_2}(y)\Big) \le n,$$

which implies that $(x,y)\in (A_1\times A_2)_{[n]}$.

Hence,
$$(A_1\times A_2)_{[n]} = A_{1_{[n]}} \times A_{2_{[n]}}.$$

Corollary 1: *Let $A_1 \in AMG(X)$ and $A_2 \in AMG(Y)$, respectively. Then for all $n\in\mathbb{N}$, $(A_1\times A_2)^{[n]} = {A_1}^{[n]} \times {A_2}^{[n]}$.*

Proof.
Follows similarly from Theorem 1 using the definition of $[n]$ and the fact that

$$m_{A_1\times A_2}\big((x,y)\big) = m_{A_1}(x) \vee m_{A_2}(y) \ge n$$

if and only if $m_{A_1}(x) \ge n$ and $m_{A_2}(y) \ge n$.

Corollary 2: *Let $A \in AMG(X)$ and $B \in AMG(Y)$, respectively. Then*
i. $(A\times B)_* = A_* \times B_*$
ii. $(A\times B)^* = A^* \times B^*$

Proof.
Follows directly from Theorem 1 by noting that

$$A_* = \bigcup_{n\in\mathbb{N}} A_{[n]},\ A^* = \bigcap_{n\in\mathbb{N}} A_* = A^{[n]},$$

and using the distributivity of Cartesian product over union and intersection:

$$(A\times B)_* = \bigcup_{n\in\mathbb{N}} (A\times B)_{[n]}$$
$$= \bigcup_{n\in\mathbb{N}} \big(A_{[n]}\times B_{[n]}\big) = A_*\times B_*$$
$$(A\times B)^* = \bigcap_{n\in\mathbb{N}} (A\times B)^{[n]}$$
$$= \bigcup_{n\in\mathbb{N}} \big(A^{[n]}\times B^{[n]}\big) = A^*\times B^*$$

Theorem 2: *Let $A_1 \in AMG(X)$ and $A_2 \in AMG(Y)$. Then $A_1\times A_2 \in AMG(X\times Y)$.*

Proof.
Let $x = (x_1, x_2),\ y = (y_1, y_2) \in X\times Y$. Then:

$$m_{A_1\times A_2}(xy) = m_{A_1\times A_2}\big((x_1y_1, x_2y_2)\big)$$
$$= m_{A_1\times A_2}(x_1y_1) \vee m_{A_1\times A_2}(x_2y_2)$$
$$\le \Big(m_{A_1}(x_1) \vee m_{A_1}(y_1)\Big) \vee \Big(m_{A_2}(x_2) \vee m_{A_2}(y_2)\Big)$$
$$= \Big(m_{A_1}(x_1) \vee m_{A_2}(x_2)\Big) \vee \Big(m_{A_1}(y_1) \vee m_{A_2}(y_2)\Big)$$
$$= m_{A_1\times A_2}(x) \vee m_{A_1\times A_2}(y),$$

using the antimultigroup property of A_1 and A_2, and the definition:

$$m_{A_1\times A_2}\big((x_1,x_2)\big) = m_{A_1}(x_1) \vee m_{A_2}(x_2)$$

Next, we check the inverse condition:

$m_{A_1\times A_2}(x^{-1}) = m_{A_1\times A_2}((x_1,x_2)^{-1})$
$= m_{A_1\times A_2}\big(({x_1}^{-1}, {x_2}^{-1})\big)$
$= m_{A_1}({x_1}^{-1}) \vee m_{A_2}({x_2}^{-1})$
$= m_{A_1}(x_1) \vee m_{A_2}(x_2)$

$= m_{A_1\times A_2}(x),$
where we used the antimultigroup inverse condition for A_1 and A_2. Thus, $A_1 \times A_2 \in AMG(X \times Y)$.

Remark 3:
The proof of Theorem 2 relies critically on the preservation of multiplicity structure under the direct product. In particular, the identity

$$m_{A_1\times A_2}\big((x_1,x_2)\big) = m_{A_1\times A_2}\big((x_1,x_2)^{-1}\big)$$

combined with the truncation identity in Theorem 1:

$$(A_1 \times A_2)_{[n]} = A_{1[n]} \times A_{2[n]}$$

Ensures that both the closure and inverse conditions required for an antimultigroup are satisfied in the product structure. Hence, Theorem 1 provides structural groundwork for Theorem 2.

Corollary 3: *Let $A_1, B_1 \in AMG(X_1)$ and $A_2, B_2 \in AMG(X_2)$ such that $A_1 \subseteq B_1$ and $A_2 \subseteq B_2$. If A_1 is a normal sub-antimultigroup of B_1 and A_2 is a normal sub-antimultigroup of B_2 then $A_1 \times A_2$ respectively such that $A_1 \subseteq B_1$ and $A_2 \subseteq B_2$. If A_1 and A_2 are normal sub-antimultigroup of B_1 and B_2, then $A_1 \times A_2$ is a normal sub-antimultigroup of $B_1 \times B_2$.*
Proof.
From theorem 2, the direct product $A_1 \times A_2$ is an antimultigroup of $X_1 \times X_2$, and so is $B_1 \times B_2$. Since $A_1 \subseteq B_1$ and $A_2 \subseteq B_2$, it follows that

$$A_1 \times A_2 \subseteq B_1 \times B_2$$

To show that $A_1 \times A_2$ is a normal sub-antimultigroup of $B_1 \times B_2$, let $(x_1,x_2)(y_1,y_2) \in X_1 \times X_2$. Then:

$$m_{A_1\times A_2}\big((x_1,x_2)(y_1,y_2)\big) = m_{A_1\times A_2}\big((x_1y_1, x_2y_2)\big)$$
$$= m_{A_1}(x_1y_1) \vee m_{A_2}(x_2y_2)$$
$$= m_{A_1}(y_1x_1) \vee m_{A_2}(y_2x_2) \text{ (since} A_1 \trianglelefteq B_1, A_2 \trianglelefteq B_2$$
$$= m_{A_1\times A_2}\big((y_1x_1, y_2x_2)\big)$$
$$= m_{A_1\times A_2}\big((y_1,y_2)(x_1,x_2)\big)$$

Thus,

$$m_{A_1\times A_2}\big((x_1,x_2)(y_1,y_2)\big) = m_{A_1\times A_2}\big((y_1,y_2)(x_1,x_2)\big)$$

which means $A_1 \times A_2$ satisfies the *normality condition* in $B_1 \times B_2$. Therefore, by Proposition 6
$A_1 \times A_2 \trianglelefteq B_1 \times B_2$.

Theorem 3: *Let A_1 and A_2 be antimultigroups of X and Y, respectively. Then*

i. $(A_1 \times A_2)_*$ is a sub-antimultigroup of $X \times Y$.
ii. $(A_1 \times A_2)^*$ is a sub-antimultigroup of $X \times Y$.
iii. $(A_1 \times A_2)_{[n]}, n \in \mathbb{N}$ is a sub-antimultigroup of $X \times Y, \forall\, n \geq m_{A_1\times A_2}(e,e')$.
iv. $(A_1 \times A_2)^{[n]}, n \in \mathbb{N}$ is a sub-antimultigroup of $X \times Y, \forall\, n \leq m_{A_1\times A_2}(e,e')$.

Proof.
From Theorem 2, the direct product $A_1 \times A_2$ is an antimultigroup of $X \times Y$. Therefore the associated multiplicity function satisfies the antimultigroup axioms on the product group $X \times Y$.
Now we apply the following results:

- From Proposition 5, for any antimultigroup $A \in AMG(G)$, the A_* and A^* are sub-antigroups of G.
- From Proposition 4, for any $n \geq m_A(e)$, the level set $A_{[n]}$ is a sub-antigroup.
- From Proposition 4, for any $n \leq m_A(e)$, the upper level set $A^{[n]}$ is a sub-antigroup

Applying these directly to $A_1 \times A_2 \in AMG(X \times Y)$, we conclude:

- $(A_1 \times A_2)_*$ and $(A_1 \times A_2)^*$ are sub-antigroups of $X \times Y$.
- $(A_1 \times A_2)_{[n]} \subseteq X \times Y$ is a sub-antigroup for all $n \geq m_{A_1\times A_2}(e,e')$.

$(A_1 \times A_2)^{[n]} \subseteq X \times Y$ is a sub-antigroup for all $n \leq m_{A_1\times A_2}(e,e')$.

RESULTS AND DISCUSSION

Union and Intersection of Cuts of Direct Product of Antimultigroup

Proposition 9: *Let $C = A_1 \times A_2$ and $D = B_1 \times B_2$, where $C, D \in AMG(X \times Y)$ and $n \in \mathbb{N}$. Then:*
i. $(C \cap D)_{[n]} = C_{[n]} \cap D_{[n]}$.
ii. $(C \cup D)_{[n]} = C_{[n]} \cup D_{[n]}$.

Proof.
(i) $C, D \in AMG(X \times Y)$, we have $C \cap D \subseteq C$ and $C \cap D \subseteq D$. By Proposition 10 for any sub-antimultigroup $A \subseteq B$, it follows that $A_{[n]} = B_{[n]}$. Therefore,

$$(C \cup D)_{[n]} = C_{[n]} \text{ and } (C \cup D)_{[n]} = D_{[n]}$$
$$\Longrightarrow (C \cap D)_{[n]} \subseteq C_{[n]} \cap D_{[n]}.$$

Now let $(x,y) \in C_{[n]} \cap D_{[n]}$, then

$$m_C(x,y) \leq n \text{ and} m_D(x,y) \leq n.$$

By definition of intersection of multiplicities:

$$m_{C\cap D}(x,y) = m_C(x,y) \wedge m_D(x,y) \leq n.$$

Hence, $(x,y) \in (C \cap D)_{[n]}$ and so

$$C_{[n]} \cap D_{[n]} \subseteq (C \cap D)_{[n]}.$$

Thus,

$$(C \cap D)_{[n]} = C_{[n]} \cap D_{[n]}.$$

(ii) Since $C \subseteq C \cup D$ and $D \subseteq C \cup D$, again by Proposition 10, we get:

$$C_{[n]} \subseteq (C \cup D)_{[n]} \text{ and } D_{[n]} \subseteq (C \cup D)_{[n]} \Longrightarrow C_{[n]} \cup D_{[n]} \subseteq (C \cup D)_{[n]}.$$

Now take $(x, y) \in (C \cup D)_{[n]}$. Then

$$m_{C \cup D}(x, y) = m_C(x, y) \vee m_D(x, y) \leq n \Longrightarrow m_C(x, y) \leq n \text{ or} m_D(x, y) \leq n.$$

So $(x, y) \in C_{[n]}$ or $(x, y) \in D_{[n]} \Longrightarrow (x, y) \in C_{[n]} \cup D_{[n]}$.

Therefore,

$$(C \cup D)_{[n]} \subseteq C_{[n]} \cup D_{[n]}.$$

Combining both inclusions:

$$(C \cup D)_{[n]} = C_{[n]} \cup D_{[n]}.$$

Proposition 10*: Let $C = A_1 \times A_2$ and $D = B_1 \times B_2$ such that $C, D \in AMG(X \times Y)$. If $C_{[n]}$ and $D_{[n]}$ are sub-antimultigroups of $X \times Y$, then $(C \cap D)_{[n]}$ is also a sub-antimultigroup of $X \times Y$.*

Proof.
Let (e, e') denote the identity in $X \times Y$. Since $(e, e') \in C_{[n]}$ and $(e, e') \in D_{[n]}$, by the sub-antimultigroup property and the definition of the cuts we have:

$$m_C\Big((e, e')\Big) \leq n \text{ and} m_D\Big((e, e')\Big) \leq n.$$

Moreover, for all $(x, y) \in X \times Y$, the values $m_C\big((x, y)\big) \leq n$ and$m_D\big((x, y)\big) \leq n$ hold on these cuts. Recall that by assumption $C_{[n]}$ and $D_{[n]}$ are sub-antimultigroups of $X \times Y$. Hence, their intersection

$$C_{[n]} \cap D_{[n]}$$

Is also a sub-antimultigroup of $X \times Y$ since the intersection of sub-antimultigroups is a sub-antimultigroup. By Proposition 9, we know that

$$(C \cap D)_{[n]} = C_{[n]} \cap D_{[n]}.$$

Therefore,

$$(C \cap D)_{[n]}$$

is a sub-antimultigroup of $X \times Y$.

Corollary 4: *Let $C = A_1 \times A_2$ and $D = B_1 \times B_2$ such that $C, D \in AMG(X \times Y)$. If $C_{[n]}$ and $D_{[n]}$ are sub-antimultigroups of $X \times Y$, then $(C \cup D)_{[n]}$ is a sub-antimultigroup of $X \times Y$ **provided that** $C \subseteq D$.*

Proof.
By Theorem 2, both $C_{[n]}$ and $D_{[n]}$ are sub-antimultigroups of $X \times Y$. Assume $C \subseteq D$. Then, by Proposition 8, we have

$$C_{[n]} \subseteq D_{[n]}.$$

Therefore, the union of these cuts simplifies to

$$C_{[n]} \cup D_{[n]} = D_{[n]}.$$

Since $D_{[n]}$ is a sub-antimultigroup of $X \times Y$, it follows that

$$C_{[n]} \cup D_{[n]}$$

Is a sub-antimultigroup of $X \times Y$.

From Proposition 9, we know that

$$(C \cup D)_{[n]} = C_{[n]} \cup D_{[n]}$$

Hence,

$$(C \cup D)_{[n]}$$

is a sub-antimultigroup of $X \times Y$.

CONCLUSION

This paper has developed the theory of direct products in the context of antimultigroups, thereby extending the classical structure from group theory to multiset-based antimultigroups. We have introduced the direct product of antimultigroups and demonstrated that the direct product of two antimultigroups is itself an antimultigroup.
In addition to the results, we have examined sub-antimultigroups induced by various types of cuts—strong and weak, upper and lower—and investigated their preservation under union and intersection. We further explored the behavior of root sets in product spaces by showing how their properties are inherited or transformed under product operations.
With this contributions, the foundational aspects of antimultigroup theory is solidified and some promising research directions are opened up. One such direction is the study of homomorphisms in the setting of direct products, including characterizations of kernel structures, image preservation, and the conditions under which factor antimultigroups may be recovered. Another prospective area involves embedding of antimultigroups, thereby linking them to broader algebraic structures.

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